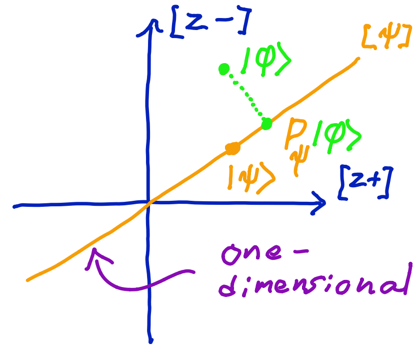
Projectors

Hermitian P s.t. $P^2 = P$



e.g. dyad $P_\psi = |\psi\rangle\langle\psi| \equiv [\psi]$

normalized $\|\psi\|^2 = 1$

check: $P_\psi^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = (\cancel{\langle\psi|\psi\rangle}^{\,1})|\psi\rangle\langle\psi| = P_\psi$ ✓

Invariant subspace: $P_\psi|\psi\rangle = (|\psi\rangle\langle\psi|)|\psi\rangle = (\cancel{\langle\psi|\psi\rangle}^{\,1})|\psi\rangle$

Spectral decomposition: let $\{|\alpha_i\rangle\}$ be eigenvectors of A (normal)

$A|\alpha_i\rangle = a_i|\alpha_i\rangle$: Define $P_i = |\alpha_i\rangle\langle\alpha_i|$: Note $P_i P_j = \delta_{ij} P_j$

Then $A = \sum_i a_i P_i = \sum_a a P_a$ check: $A|\alpha_i\rangle = a_i|\alpha_i\rangle$ $\forall$ eigenvectors

$\Rightarrow$ acts as A for all $|\psi\rangle \in \mathcal{H}$

Examples:

- Identity: $I = \sum_i 1 \cdot P_i$ "Decomposition of the identity"
- spin-$\frac{1}{2}$: $I = [z+] + [z-]$, $S_z = (\frac{\hbar}{2})[z+] + (\frac{-\hbar}{2})[z-]$

  $|x+\rangle = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle)$

  $[x+] = \frac{1}{2}([z+] + [z-] + |z+\rangle\langle z-| + |z-\rangle\langle z+|)$

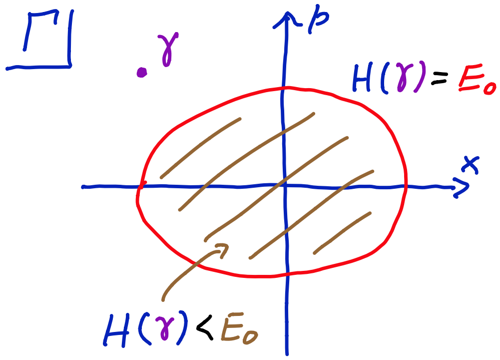
- Harmonic oscillator:

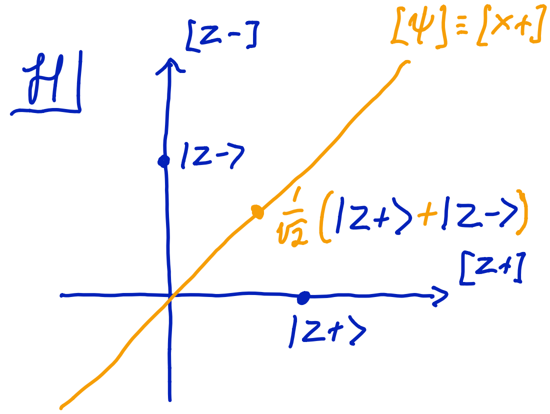
  $H|n\rangle = E_n|n\rangle$ $E_n = (n+\frac{1}{2})\hbar\omega$ $n = 0, 1, 2, \cdots$

  $I = \sum_{n=0}^{\infty} |n\rangle\langle n|$

  $H = \sum_{n=0}^{\infty} E_n [n]$

Physical Variables





Classical variable:

real-valued function of γ

e.g. $H(\gamma)$, x, p, etc.

Quantum variable:

value of Hermitian operator

e.g. H, S_z, S_x, etc.

Spectral decomposition: $H = \sum_j E_j P_j = \sum_E E P_E$

Within subspace P_E, H "has value" E. H acts like real # E.

$H|\psi\rangle = E|\psi\rangle \quad \forall |\psi\rangle$ s.t. $P_E|\psi\rangle = |\psi\rangle$

Outside subspace of any P_E, H does not "have value"

$H|\psi\rangle \neq E|\psi\rangle$ for any E: value of H not defined

Example: S_z has value $\frac{\hbar}{2}$ for $|\psi\rangle = c|z+\rangle$

" " $-\frac{\hbar}{2}$ " " $c|z-\rangle$

S_z has no value for $|\psi\rangle = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle) = |x+\rangle$

undefined

Physical Property

classical subset of Γ

example: "energy $< E_0$"

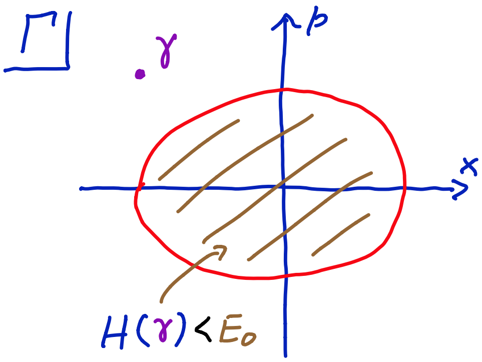
$\{\gamma \in \Gamma$ s.t. $H(\gamma) < E_0\}$

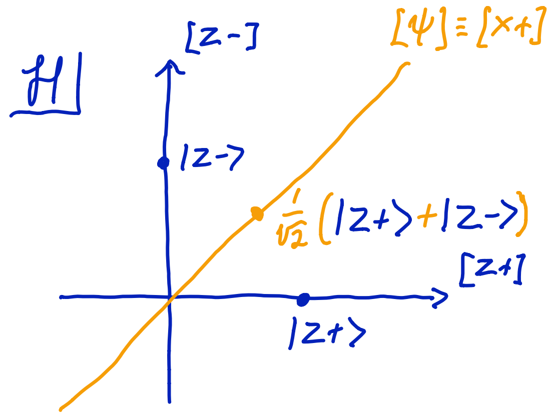
Quantum linear subspace of $\mathcal{H}$

example: "$S_x = +\frac{\hbar}{2}$"

property P_ψ with $|\psi\rangle = |x+\rangle$

Summary





Physical variable $H(\gamma)$

Physical property $P \subset \Gamma$

$P = \{\gamma \in \Gamma \text{ s.t. } H(\gamma) < E_0\}$

Indicator function

$$P(\gamma) = \begin{cases} 1 & H(\gamma) < E_0 \\ 0 & \text{else} \end{cases}$$

Physical variable S_x

Physical property $P \subset \mathcal{H}$

Linear* subspace where $S_x = \frac{\hbar}{2}$

$P = \{c|x+\rangle : c \in \mathbb{C}\}$

Projector $P_{x+} = [x+]$

* dimension 1 or greater