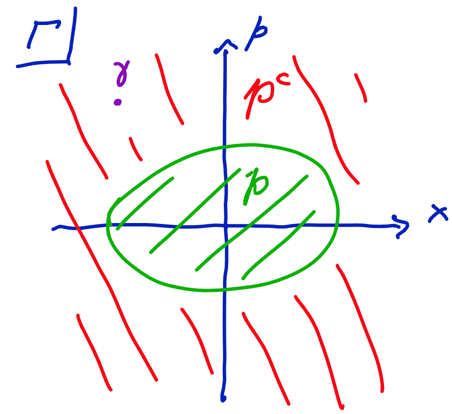
Classical Logic



Property $p \subset \Gamma$ is subset

Indicator function $P(\gamma) = \begin{cases} 1 & \gamma \in p \\ 0 & \text{else} \end{cases}$

Logical Operation	Set Theoretic	Algebraic	Example
Property p	$p \subset \Gamma$	$P(\gamma)$	$H(\gamma) \leq E_0$
Negation $\neg p$	$p^c = \Gamma - p$	$P^c(\gamma) = 1 - P(\gamma)$	$H(\gamma) > E_0$
Conjunction $p \wedge Q$	$p \cap Q$	$P(\gamma)Q(\gamma)$	$H(\gamma) \leq E_0$ and $x > 0$
Disjunction $p \vee Q$	$p \cup Q$	$P + Q - PQ$	$H(\gamma) \leq E_0$ or $x > 0$

Identities:

$$\neg(p \cap Q) = (\neg p) \cup (\neg Q)$$

$$\neg(p \cup Q) = (\neg p) \cap (\neg Q)$$

Distributive Law:

$$p \cup (Q \cap R) = (p \cup Q) \cap (p \cup R)$$

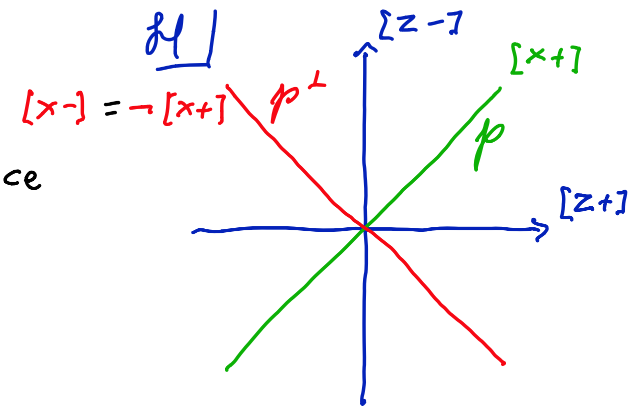
$$p \cap (Q \cup R) = (p \cap Q) \cup (p \cap R)$$

Boolean Algebra: $\{p, Q, R \ldots\}$ that is closed under $\neg, \wedge, \vee$

Example: $\{\text{True}, \text{False}, p, \neg p\}$ e.g. $p \vee \neg p = \text{True}$ *

* Apologies to Shakespeare

Quantum Logic



Property $\mathcal{P} \subset \mathcal{H}$ is linear subspace

Projector P

Logical Operation	Set Theoretic	Algebraic	Example
Property $\mathcal{P}$	$\mathcal{P} \subset \mathcal{H}$ (linear)	$P: \mathcal{H} \to \mathcal{P}$	$S_x = \frac{\hbar}{2}$
$\neg \mathcal{P}$	$\mathcal{P}^\perp = \{ \lvert\psi\rangle \in \mathcal{H} : P\lvert\psi\rangle = 0 \}^{*}$	$I - P^{**}: \mathcal{H} \to \mathcal{P}^\perp$	$\neg\left(S_x = \frac{\hbar}{2}\right) = \left(S_x = -\frac{\hbar}{2}\right)$
$\mathcal{P} \wedge \mathcal{Q}$	$\mathcal{P} \cap \mathcal{Q}\ ?^{***}$	$PQ\ ?^{****}$	$[x+] \wedge [z+]$ true or false
$\mathcal{P} \vee \mathcal{Q}$	$\mathcal{P} \cup \mathcal{Q}\ ?^{*****}$	$P + Q - PQ\ ?$	

* $\mathcal{P}^c = \mathcal{H} - \mathcal{P}$ is not a linear subspace

e.g. $|z+\rangle,\ |z-\rangle \in \mathcal{P}^c$ but $|x+\rangle = \frac{1}{\sqrt{2}}\left(|z+\rangle + |z-\rangle\right) \in \mathcal{P}$

** Is $I - P$ a projector? $(I-P)^2 = I^2 - IP - PI + P^2 = I - P$ ✓

$(I-P)|x+\rangle = |x+\rangle - |x+\rangle = |0\rangle$ "False" ✓

*** Should $(\mathcal{P} = [x+])$ and $(\mathcal{Q} = [z+])$ be False? Note: $\mathcal{P} \cap \mathcal{Q} = |0\rangle$

**** What if $PQ \neq QP$?

***** $\mathcal{P} \cup (\neg \mathcal{P})$ does not cover $\mathcal{H}$!

What could go wrong?

$$\text{True} = \text{True} \wedge \text{True} = ([z+] \vee [z-]) \wedge ([x+] \vee [x-])$$

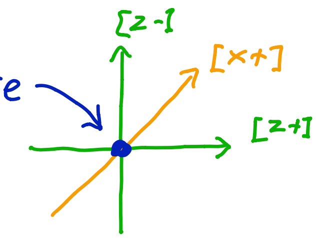
distributive law $\longrightarrow$ $= ([z+] \wedge [x+]) \vee ([z+] \wedge [x-]) \vee ([z-] \wedge [x+]) \vee ([z-] \wedge [x-])$

$$= \text{False} \vee \text{False} \vee \text{False} \vee \text{False}$$

$$= \text{False}$$

$[z+] \wedge [x+]$

$= [0]$ False



Resolution

Only discuss joint properties P and Q if $PQ = QP$

Check: is PQ a projector (property)?

$$(PQ)^2 = PQPQ = P^2 Q^2 = PQ \checkmark$$

$P + Q - PQ$ is also a projector $\checkmark$