Probability

State $|\psi\rangle$ property $\mathcal{P}$ projector P

Conditional probability $0 \leq Pr(P \,|\, |\psi\rangle) \leq 1$ $Pr = ?$

1. If $|\psi\rangle \in \mathcal{P}$ i.e. $P|\psi\rangle = |\psi\rangle$ then $Pr = 1$

2. If $|\psi\rangle \in \neg\mathcal{P} = \mathcal{P}^\perp$ i.e. $P|\psi\rangle = |0\rangle$ then $Pr(P\,|\,|\psi\rangle) = 0$

   $P^\perp|\psi\rangle = |\psi\rangle$ $Pr(\neg P\,|\,|\psi\rangle) = 1$

The rest of $\mathcal{H}$? $\Rightarrow$ Born Rule

assuming $\|\psi\|^2 = \langle\psi|\psi\rangle = 1$

$$Pr(P\,|\,|\psi\rangle) = \|P|\psi\rangle\|^2 = (\langle\psi|P^\dagger)(P|\psi\rangle) = \langle\psi|P|\psi\rangle$$

expectation value of P in state $|\psi\rangle$

Why? See online notes, Gleason's theorem

Average value of physical variable $A = \sum_a a P_a$

$$\langle A\rangle_\psi = \sum_a a\, Pr(A = a\,|\,\psi) = \sum_a a \|P_a|\psi\rangle\|^2 = \sum_a a \langle\psi|P_a|\psi\rangle = \langle\psi|A|\psi\rangle$$

expectation value of A in state $|\psi\rangle$

Density Operator

ket Projector Pure state density operator

$|\psi\rangle \longrightarrow [\psi] = |\psi\rangle\langle\psi| \longrightarrow \rho = [\psi]$

$c|\psi\rangle \longleftarrow [\psi] \longleftarrow \rho$

$|c|^2 = 1$

Probability

$$\Pr(P|\psi) = \langle\psi|P|\psi\rangle = \langle\psi|P\underbrace{\sum_i |i\rangle\langle i|}_{I}\psi\rangle$$

$$= \sum_i \langle i|\psi\rangle\langle\psi|P|i\rangle = \mathrm{Tr}(\rho P)$$

Average using density operator

$$\langle A\rangle_\rho = \langle\psi|A|\psi\rangle = \langle\psi|A\sum_i |i\rangle\langle i|\psi\rangle$$

$$= \sum_i \langle i|\psi\rangle\langle\psi|A|i\rangle = \mathrm{Tr}(\rho A)$$

Mixed State

Let $\{P_j\}$ be orthogonal 1-D projectors (pure states)

Set $\rho = \sum_j p_j P_j$ with $\sum_j p_j = 1$ (probabilities)

Note: $\mathrm{Tr}(\rho) = \sum_j p_j \,\mathrm{Tr}\, P_j = \sum_j p_j = 1$ (eigenvalues of $\rho \geq 0$)

Consider $\mathrm{Tr}(\rho^2) = \sum_i p_i \sum_j p_j \,\mathrm{Tr}(P_i P_j) = \sum_j p_j^2 \leq 1$

$\mathrm{Tr}(\rho^2) = 1$ iff $p_j = \begin{cases} 1 & \text{for one } j \\ 0 & \text{all other } j \end{cases}$

$\therefore$ $\mathrm{Tr}(\rho^2) = 1 \Leftrightarrow$ pure state $\quad \mathrm{Tr}(\rho^2) < 1 \Leftrightarrow$ mixed

Average in mixed state

$$\langle A\rangle_\rho = \sum_i p_i \langle A\rangle_{P_i} = \sum_i p_i \,\mathrm{Tr}(P_i A) = \mathrm{Tr}(\rho A)$$

Matrix representations

$|x+\rangle = \frac{1}{\sqrt{2}}\left(|z+\rangle + |z-\rangle\right)$

$$\rho = |x+\rangle\langle x+| = \frac{1}{2}\left(|z+\rangle\langle z+| + |z+\rangle\langle z-| + |z-\rangle\langle z+| + |z+\rangle\langle z-|\right)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ in basis } \{|z+\rangle, |z-\rangle\}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ in basis } \{|x+\rangle, |x-\rangle\}$$

What if we aren't sure about $|\psi\rangle$?

e.g. 75% sure $S_x = +\frac{\hbar}{2}$ 25% sure $S_x = -\frac{\hbar}{2}$

$\Rightarrow$ Mixed state $\rho = \frac{3}{4}\rho_{x+} + \frac{1}{4}\rho_{x-}$

$$= \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix} \text{ or } = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}$$

$x\pm$ basis $\qquad$ $z\pm$ basis

$$Pr([z+]|\rho) = Tr\left[\begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \frac{1}{2}$$ used $z\pm$ basis

$$Pr([x+]|\rho) = Tr\left[\begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \frac{3}{4}$$ used $x\pm$ basis