Composite Systems and Tensor Products

e.g. electron (spatial position + spin)
hydrogen atom (electron + proton)

Classical Mechanics

$(x,p) \in \Gamma \quad \dim(\Gamma) = 2 \quad$ (1 particle)

2 particles: $(x_a\, p_a) \in \Gamma_a \quad (x_b\, p_b) \in \Gamma_b \Rightarrow (x_a\, p_a\,;\, x_b\, p_b) \in \Gamma_a \times \Gamma_b$

$\dim(\Gamma_a \times \Gamma_b) = \dim(\Gamma_a) + \dim(\Gamma_b)$

"x" is Cartesian product

Quantum Mechanics

$\mathcal{H}_a$ and $\mathcal{H}_b \to \mathcal{H}_{ab} = \mathcal{H}_a \otimes \mathcal{H}_b$

$\otimes$ is tensor product

Example: given $|a\rangle \in \mathcal{H}_a,\ |b\rangle \in \mathcal{H}_b \Rightarrow \underbrace{|a\rangle \otimes |b\rangle}_{\text{product state}} \in \mathcal{H}_{ab}$

But: $\exists$ many non-product states!!

Basis for $\mathcal{H}_{ab}$:

Let $\{|a_j\rangle\}$ be a basis for $\mathcal{H}_a$
" $\{|b_k\rangle\}$ " $\mathcal{H}_b$

can take $\{|a_j\rangle \otimes |b_k\rangle\}$ as basis for $\mathcal{H}_{ab}$

Notation: $|a_j\rangle \otimes |b_k\rangle \to |j\rangle_a \otimes |k\rangle_b \to |j\rangle_a |k\rangle_b \to |j\,k\rangle$

a 1st, b 2nd

Example: Two spin-1/2 basis $\{|z\pm\rangle_a\} \equiv \{|\pm\rangle_a\}$ (Same for b)

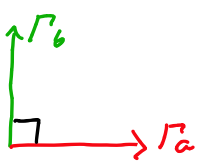
Tensor product basis: $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$

e.g. $|z+\rangle_a \otimes |z-\rangle_b$

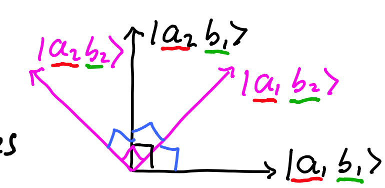
General state: $\sum_{j,k=\pm} \Psi_{jk} |jk\rangle$

Note: $\dim(\mathcal{H}_{ab}) = \dim(\mathcal{H}_a) \cdot \dim(\mathcal{H}_b) \gg \dim(\mathcal{H}_a) + \dim(\mathcal{H}_b)$ (usually)

Cartesian: add bases



Tensor: multiply bases



Example

Spinor = particle with spin and position

$$\mathcal{H} = (\text{Space of spinless 3D wavefunctions}) \otimes (\text{Space of spin-}1/2)$$

$$\{\psi(\vec{r})\} \qquad\qquad \{\alpha|+\rangle + \beta|-\rangle\}$$

Product state $(\alpha|+\rangle + \beta|-\rangle) \otimes \psi(\vec{r})$

General state $|w\rangle = \sum_j (\alpha_j|+\rangle + \beta_j|-\rangle) \otimes \psi_j(\vec{r}) = \begin{pmatrix} w_+(\vec{r}) \\ w_-(\vec{r}) \end{pmatrix} \begin{matrix} (\text{spin} \uparrow) \\ (\text{spin} \downarrow) \end{matrix}$

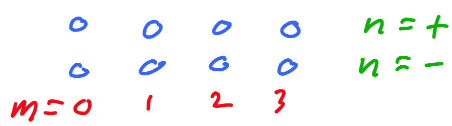
$$w_+(\vec{r}) = \langle \vec{r}+|w\rangle = \sum_j \alpha_j \psi_j(\vec{r})$$

$$w_-(\vec{r}) = \langle \vec{r}-|w\rangle = \sum_j \beta_j \psi_j(\vec{r})$$

Toy models

(discrete, finite dimensional, aid clarity)

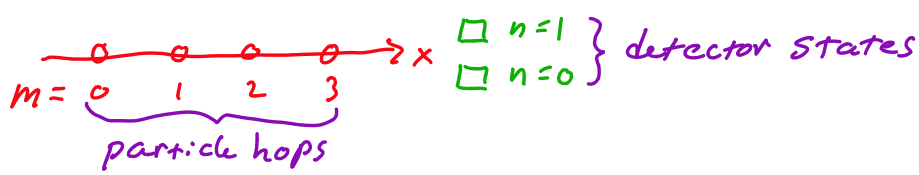
a) particle with spin



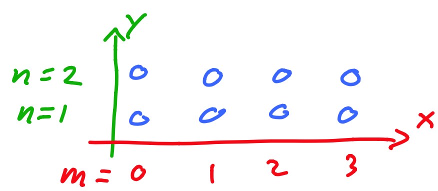
$$\mathcal{H}_{mn} = \mathcal{H}_m \otimes \mathcal{H}_n$$

$$\dim \mathcal{H}_{mn} = 4 \cdot 2 = 8$$

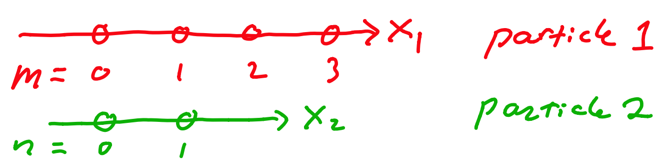
b) Particle & detector



c) Particle in 2D



d) Two particles



All are isomorphic! $\quad \psi_n(m) \equiv \langle mn|\psi\rangle \quad |\psi\rangle \in \mathcal{H}_{mn}$

Entangled States

Expanding any $|\psi\rangle \in \mathcal{H}_{ab}$ in $\{|j_a\rangle \otimes |k_b\rangle\}$:

$|\psi\rangle = \sum_{jk} \Psi_{jk} |j\,k\rangle$ $\quad$ Ψ_{jk} is $(\dim \mathcal{H}_a) \times (\dim \mathcal{H}_b)$ rectangular array

$\mathrm{Rank}(\Psi) = \#$ independent rows $= \#$ independent columns

If $\mathrm{Rank}(\Psi) = 1$ then $\exists\, |a\rangle \in \mathcal{H}_a,\ |b\rangle \in \mathcal{H}_b$ s.t. $|\psi\rangle = |a\rangle \otimes |b\rangle$
product state!

If $\mathrm{Rank}(\Psi) > 1$ then $|\psi\rangle$ is "entangled state"

Example two spin-1/2's:

$$\left.\begin{aligned} |a\rangle &= \alpha_+ |+\rangle + \alpha_- |-\rangle \\ |b\rangle &= \beta_+ |+\rangle + \beta_- |-\rangle \end{aligned}\right\}$$ most general elements of $\mathcal{H}_a$ and $\mathcal{H}_b$

Most general product state:

$$|\psi\rangle = |a\rangle \otimes |b\rangle = \underbrace{\alpha_+\beta_+}_{\Psi_{++}} |{+}{+}\rangle + \underbrace{\alpha_+\beta_-}_{\Psi_{+-}} |{+}{-}\rangle + \underbrace{\alpha_-\beta_+}_{\Psi_{-+}} |{-}{+}\rangle + \underbrace{\alpha_-\beta_-}_{\Psi_{--}} |{-}{-}\rangle$$

Note: $\Psi_{++}\Psi_{--} = \Psi_{+-}\Psi_{-+}$ (i.e. $\det \Psi = 0$) is necessary and sufficient for $|\psi\rangle$ to be product state of two spin-1/2's

Entangled state: $|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle - |{-}{+}\rangle\right)$ "EPR state"

Check: $\Psi_{++} = \Psi_{--} = 0 \quad \Psi_{+-} = -\Psi_{-+} = \frac{1}{\sqrt{2}} \quad 0 \cdot 0 \neq \frac{1}{\sqrt{2}} \cdot \frac{-1}{\sqrt{2}}$

$\therefore$ not a product state