Recall: product state $|a\rangle \otimes |b\rangle = |ab\rangle \in \mathcal{H}_a \otimes \mathcal{H}_b$

Adjoint: $(|a\rangle \otimes |b\rangle)^\dagger = (|a\rangle)^\dagger \otimes (|b\rangle)^\dagger = \langle a| \otimes \langle b|$

do not swap a & b!

Inner product: $\left.\begin{array}{l} |\psi\rangle = |a\rangle \otimes |b\rangle \\ |\psi'\rangle = |a'\rangle \otimes |b'\rangle \end{array}\right\} \langle \psi|\psi'\rangle = \langle a|a'\rangle \cdot \langle b|b'\rangle$

Operators: A, B act on $\mathcal{H}_a, \mathcal{H}_b \Rightarrow A \otimes B$ acts on $\mathcal{H}_{ab}$

$(A \otimes B)(|a\rangle \otimes |b\rangle) = (A|a\rangle) \otimes (B|b\rangle)$

sum of products $\sum_{jk} A_j \otimes B_k$

product of products $(A \otimes B)(A' \otimes B') = (AA') \otimes (BB')$

Example: $(A \otimes I_b)(I_a \otimes B) = A \otimes B = AB$ ← potentially confusing - acts on $\mathcal{H}_a \otimes \mathcal{H}_b$

Example: Two spin 1/2's

$A = [z+]_a = |+\rangle_a \langle +|_a \quad , \quad I_b = |+\rangle_b \langle +|_b + |-\rangle_b \langle -|_b$

$A \otimes I_b = |++\rangle\langle ++| + |+-\rangle\langle +-| \equiv [z_a+]$ $\otimes I_b$ is implicit

"Spin a has $S_z = +\hbar/2$"

Check: $(A \times I_b)^2 = A \times I_b$ because $\langle ++|+-\rangle = \langle +-|++\rangle = 0$ ✓

Meaning of EPR state: $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|z+z-\rangle - |z-z+\rangle)$

$[\psi_0] = |\psi_0\rangle\langle\psi_0| = \frac{1}{2}\left(|+-\rangle\langle +-| - |+-\rangle\langle -+| - |-+\rangle\langle +-| + |-+\rangle\langle -+|\right)$

What are properties of spin a when $[\psi_0]$ is true?

$\left.\begin{array}{l} [z_a+][\psi_0] = \frac{1}{2}\left(|+-\rangle\langle +-| - |+-\rangle\langle -+|\right) \\ [\psi_0][z_a+] = \frac{1}{2}\left(|+-\rangle\langle +-| - |-+\rangle\langle +-|\right) \end{array}\right\}$ non-commuting: cannot assign value to z_a! Or to z_b or to x_a, ... !

Let $P \equiv [++] + [--]$ "both spins the same"

$Q \equiv [+-] + [-+]$ "both spins different"

$P[\psi_0] = [\psi_0]P = 0$ "both same" is false

$Q[\psi_0] = [\psi_0]Q = [\psi_0]$ "both different" is true

$Q = I - P$ "not both same" is true

Logical framework for reasoning: Choose decomposition of the identity, e.g. $\{P, [\psi_0], Q - [\psi_0]\}$ orthogonal projectors summing to I. Can make statements about truth/falsehood within event algebra of framework, e.g. "both same" is false. Properties of spin a or b alone, e.g. $[z_a +]$, lie outside framework and cannot be discussed.