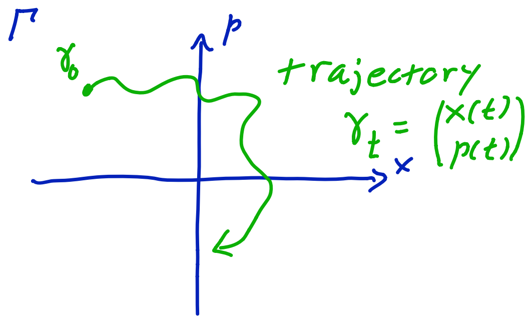
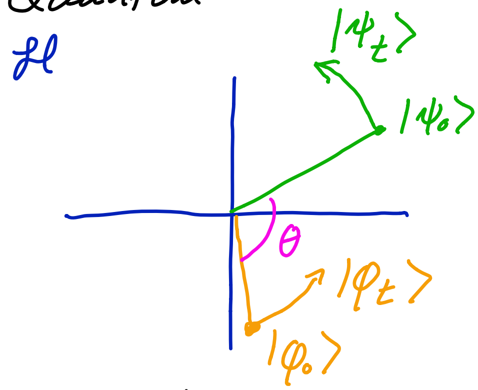
Unitary time dependence

Classical



Quantum



Hamilton's Equation
(symplectic form)

$$\frac{d}{dt}\vec{\gamma} = J\,\frac{\partial H}{\partial \vec{\gamma}} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Schrödinger's Equation

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

$$H = H(t)$$

Unitarity initial states $|\psi_0\rangle \to |\psi_t\rangle$, $|\varphi_0\rangle \to |\varphi_t\rangle$

$$\frac{d}{dt}\langle\varphi_t|\psi_t\rangle = \langle\varphi_t|\left(\frac{d}{dt}|\psi_t\rangle\right) + \left(\frac{d}{dt}\langle\varphi_t|\right)|\psi_t\rangle \qquad \left(\frac{d}{dt}|\varphi_t\rangle\right)^{\dagger}$$

$$= \langle\varphi_t|\left(\frac{-i}{\hbar}H|\psi_t\rangle\right) + \left(\frac{-i}{\hbar}H|\varphi_t\rangle\right)^{\dagger}|\psi_t\rangle$$

$$= \frac{-i}{\hbar}\langle\varphi_t|H|\psi_t\rangle + \frac{i}{\hbar}\langle\varphi_t|H^{\dagger}|\psi_t\rangle = 0$$

$$H^{\dagger} = H$$

$\therefore$ Time evolution preserves inner product

In particular it preserves norms $\|\psi_t\|^2 = \langle\psi_t|\psi_t\rangle$

Analog of classical Liouville theorem

linear transformation

Time development operator : $|\psi_t\rangle = T(t,t')|\psi_{t'}\rangle$
$|\varphi_t\rangle = T(t,t')|\varphi_{t'}\rangle$ $\Big\}$ evolve $t' \to t$ same T

classical analog is Canonical transformation $\vec{\gamma}(t') \to \vec{\gamma}(t)$ (nonlinear)

Properties of T

preservation of norm

$$\langle \varphi_t | \psi_t \rangle = \left(\langle \varphi_{t'} | T^\dagger(t,t') \right)\left(T(t,t') | \psi_{t'} \rangle \right) = \langle \varphi_{t'} | \psi_{t'} \rangle$$

$\therefore \; T^\dagger T = I$ T is unitary

Also, $T(t,t) = I$

$$T(t,t')\,T(t',t'') = T(t,t'')$$

$$T^\dagger(t,t') \underset{\text{unitarity}}{=} T^{-1}(t,t') \underset{\text{invertability}}{=} T(t',t)$$

Properties of unitary matrices   operator U basis $\{|b_j\rangle\}$

$U_{jk} \equiv \langle b_j | U | b_k \rangle, \quad U = \sum_{jk} |b_j\rangle U_{jk} \langle b_k|$

$U^\dagger U = U U^\dagger = I \Rightarrow$ all rows orthonormal and all Columns

General $U = e^{iK}$ ← Hermitian $e^{iK} = I + iK + \frac{i^2}{2} K^2 + \cdots$

General 2×2: $U = \begin{pmatrix} \alpha & \beta \\ -e^{i\varphi}\beta^* & e^{i\varphi}\beta^* \end{pmatrix}$ $|\alpha|^2 + |\beta|^2 = 1$ $\varphi \in \mathbb{R}$

Proof: orthonormal rows/colomns

Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi_t\rangle = i\hbar \frac{\partial}{\partial t} T(t,t') |\psi_{t'}\rangle = H |\psi_t\rangle = H\, T(t,t') |\psi_{t'}\rangle$$

$$\therefore \; i\hbar \frac{\partial}{\partial t} T(t,t') = H(t)\, T(t,t')$$

adjoint $\quad -i\hbar \frac{\partial}{\partial t'} T(t,t') = T(t,t')\, H(t')$

Example (H independent of t)

$T(t,t) = I$

$$\frac{1}{T}\frac{\partial T}{\partial t} = \left(\frac{-i}{\hbar}\right) H = \frac{\partial \ln T}{\partial t} \Rightarrow \ln T(t,t') = \frac{-i}{\hbar}(t-t')H$$

$$\therefore T(t,t') = T(t-t') = e^{(-i/\hbar)(t-t')H}$$

Spectral decomposition

$$H = \sum_j E_j P_j$$

$\{E_j\}$ discrete energy levels

$P_j \rightarrow E_j$ subspace

$$T(t-t') = \sum_j e^{(-i/\hbar)(t-t')E_j} P_j$$

Multiple time Born rule

$$Pr(P_t \mid \psi_{t'}) = Pr(P_t \mid \psi_t) = \| P\, T(t,t') |\psi_{t'}\rangle \|^2$$

Example: $$Pr(|\psi_t\rangle \mid |\psi_{t'}\rangle) = \| \langle \psi_t | T(t,t') | \psi_{t'}\rangle \|^2$$

$$= \| \langle \psi_t | \psi_t \rangle \|^2$$

$$= 1$$