Bloch Sphere (spin-1/2 state space)

General $|\psi\rangle = \alpha|z+\rangle + \beta|z-\rangle$

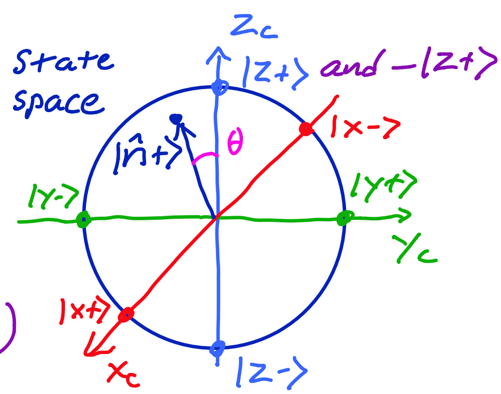
$0 \le \alpha \le 1 \Rightarrow 0 \le \theta \le \pi$

$$\left.\begin{array}{l}\text{Normalize: } |\alpha|^2 + |\beta|^2 = 1 \\ \text{Phase choice: } \alpha \geq 0 \text{ real}\end{array}\right\} |\psi\rangle = \underbrace{\cos\frac{\theta}{2}}_{\alpha}|z+\rangle + \underbrace{\sin\frac{\theta}{2}e^{i\varphi}}_{\beta}|z-\rangle$$

spherical coordinates $(\theta,\varphi) \longrightarrow$ Cartesian coordinates

$z_c \equiv \cos\theta \qquad x_c + i y_c = \sin\theta e^{i\varphi}$

$|\alpha|^2 + |\beta|^2 = z_c^2 + x_c^2 + y_c^2 = 1$



$\theta = 0, \pi \Rightarrow |z+\rangle, |z-\rangle$

$\theta = 2\pi \Rightarrow -|z+\rangle \equiv +|z+\rangle$ (Same state)

$\theta = \pi/2 : \varphi = 0, \pi \Rightarrow |x+\rangle, |x-\rangle$

$\theta = \pi/2 : \varphi = \pi/2, -\pi/2 \Rightarrow |y+\rangle, |y-\rangle$

$\therefore |x\pm\rangle = \frac{1}{\sqrt{2}}(|z+\rangle \pm |z-\rangle), \quad |y\pm\rangle = \frac{1}{\sqrt{2}}(|z+\rangle \pm i|z-\rangle)$

Pauli Matrices $S_x = \frac{\hbar}{2}\sigma_x \quad S_y = \frac{\hbar}{2}\sigma_y \quad S_z = \frac{\hbar}{2}\sigma_z$

$|x\pm\rangle$ are eigenvectors of S_x: $S_x|x\pm\rangle = \pm\frac{\hbar}{2}|x\pm\rangle$

In basis $\{|z+\rangle, |z-\rangle\}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Express any 2×2 $M = a_0 I + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z \equiv a_0 I + \vec{a}\cdot\vec{\sigma}$

$$M = \begin{pmatrix} a_0 + a_z & a_x - i a_y \\ a_x + i a_y & a_0 - a_z \end{pmatrix}$$

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

vector of matrices

Spin operator in direction $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$$S_{\hat{n}} \equiv \hat{n}\cdot\vec{S} = \frac{\hbar}{2}\hat{n}\cdot\vec{\sigma} = \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & \cos\theta\end{pmatrix}$$

State: $|\hat{n}+\rangle = \begin{pmatrix}\cos\theta/2 \\ \sin\theta/2 \, e^{i\varphi}\end{pmatrix}$

check: $S_n|\hat{n}\pm\rangle = \pm\frac{\hbar}{2}|\hat{n}\pm\rangle$ via trig. identities

Useful facts $\quad \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I, \quad \sigma_x\sigma_y = i\sigma_z \qquad \sigma_j\sigma_k = i\epsilon_{jkl}\sigma_l$

$$e^{i\theta\hat{n}\cdot\vec{\sigma}} = I + i\theta\,\hat{n}\cdot\vec{\sigma} + \frac{1}{2}(i\theta)^2(\hat{n}\cdot\vec{\sigma})^2 + \cdots$$

$$(\hat{n}\cdot\vec{\sigma})^2 = (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2$$
$$= (n_x^2 + n_y^2 + n_z^2)I$$

$$\therefore e^{i\theta\hat{n}\cdot\vec{\sigma}} = \cos\theta\, I + i\sin\theta\; \hat{n}\cdot\vec{\sigma}$$

Spin-1/2 density matrix

Recall pure state $|\psi\rangle \rightarrow \rho = |\psi\rangle\langle\psi|, \quad \mathrm{Tr}\,\rho = 1$

Mixed state $\{(|\psi_j\rangle, p_j)\} \longrightarrow \rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|, \quad \mathrm{Tr}\,\rho = \sum_j p_j = 1$

$\mathrm{Tr}\,\rho^2 = \sum_j p_j^2 \leq 1$

Most general spin-1/2 density matrix:

$\rho = \frac{1}{2}(I + \rho_x\sigma_x + \rho_y\sigma_y + \rho_z\sigma_z)$ note $\mathrm{Tr}\, I = 2 \Rightarrow \mathrm{Tr}\,\rho = 1$

Eigenvalues of ρ: $\lambda_\pm = \frac{1}{2}(1 \pm |\vec{\rho}|)$ with $\vec{\rho} = (\rho_x\, \rho_y\, \rho_z)$ $\quad |\vec{\rho}| \leq 1$

Expectation values:

$$\langle S_x \rangle = \frac{\hbar}{2}\,Tr(\rho\sigma_x) = \frac{\hbar}{2}\cdot\frac{1}{2}\,Tr(\underbrace{\sigma_x I}_{\sigma_x} + \rho_x \underbrace{\sigma_x^2}_{I} + \rho_y \underbrace{\sigma_y\sigma_x}_{-i\sigma_z} + \rho_z \underbrace{\sigma_z\sigma_x}_{\sigma_y})$$

$$= \frac{\hbar}{2}\rho_x$$

$\therefore (\rho_x, \rho_y, \rho_z)$ are components of $\langle \vec{S} \rangle$

Mixed or pure?

$$Tr\,\rho^2 = \frac{1}{4}\,Tr\Big\{ I^2 + \rho_x I\sigma_x + \rho_y I\sigma_y + \rho_z I\sigma_z$$
$$+ \rho_x\sigma_x I + \rho_x^2\sigma_x^2 + \cdots \quad \cdots$$
$$\cdots \quad \cdots + \rho_y^2\sigma_y^2 \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots + \rho_z^2\sigma_z^2 \Big\}$$
$$= \frac{1}{2}(1 + \rho_x^2 + \rho_y^2 + \rho_z^2) \leq 1 \checkmark$$

Pure state $|\vec{\rho}|^2 = 1$, $\vec{\rho} = \hat{n}$ surface

Mixed state $|\vec{\rho}| < 1$ interior