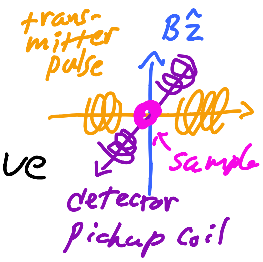
Time-dependent Hamiltonian $H = \frac{1}{2}\hbar\omega(t)\hat{n}(t)\cdot\vec{\sigma}$

Short-time approximation $T(t+\Delta t, t) = e^{-i(\omega(t)\Delta t/2)\hat{n}(t)\cdot\vec{\sigma}}$

Exactly solvable case $\omega(t)$ constant $\hat{n}(t)$ rotating (NMR)

$$\omega\hat{n}(t) = \omega_z\hat{z} + \omega_p \underbrace{(\cos\omega_r t\,\hat{x} + \sin\omega_r t\,\hat{y})}_{\text{"rotating" radio wave frequency } \omega_r}$$

〃 B_z "pulse" amplitude



Trick for solving - rotating coordinate system

Given $|\psi\rangle$ in basis $\{|b_j\rangle\}$: $|\psi\rangle = \left(\sum_j |b_j\rangle\langle b_j|\right)|\psi\rangle = \sum_j\left(\langle b_j|\psi\rangle\right)|b_j\rangle$

express in basis $\{|\bar{b}_k\rangle\}$: $\langle\bar{b}_k|\psi\rangle = \langle\bar{b}_k|\left(\sum_j |b_j\rangle\langle b_j|\right)|\psi\rangle = U_{kj}\langle b_j|\psi\rangle$

Unitary matrix $\langle\bar{b}_k|b_j\rangle$

Transformation of Coordinate system

$|\psi\rangle \to |\bar{\psi}\rangle = S(t)|\psi\rangle$ inverse $|\psi\rangle = S^\dagger|\bar{\psi}\rangle$

original basis   new basis   unitary

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_N\end{pmatrix} \qquad \begin{pmatrix}\psi_{\bar{1}}\\ \psi_{\bar{2}}\\ \vdots\\ \psi_{\bar{N}}\end{pmatrix}$$

Schrödinger $\Rightarrow$ $i\hbar\frac{d}{dt}|\bar{\psi}\rangle = i\hbar\frac{dS}{dt}|\psi\rangle + S\left(i\hbar\frac{d}{dt}|\psi\rangle\right)$

$$= (i\hbar\dot{S}S^\dagger + SHS^\dagger)|\bar{\psi}\rangle$$

$\bar{H}_1$ new term when $\dot{S} \neq 0$ $\bar{H}_0$ Original H in new coords

Set $H = \frac{1}{2}\omega_z \sigma_z + \frac{1}{2}\omega_p \left(\cos(\omega_r t)\sigma_x + i\sin(\omega_r t)\sigma_y\right)$

Note: $\hbar = 1$

Let $S(t) = \begin{pmatrix} e^{i\omega_r t/2} & 0 \\ 0 & e^{-i\omega_r t/2} \end{pmatrix}$ Frame rotates around $\hat{z}$ at frequency ω_r ... matches RF

$$\bar{H}_1 = i\dot{S}S^\dagger = i\begin{pmatrix} \frac{i\omega_r}{2} e^{i\omega_r t/2} & 0 \\ 0 & \frac{-i\omega_r}{2} e^{-i\omega_r t/2} \end{pmatrix}\begin{pmatrix} e^{-i\omega_r t/2} & 0 \\ 0 & e^{+i\omega_r t/2} \end{pmatrix}$$

$$= -\frac{1}{2}\omega_r \sigma_z$$ effective magnetic field // $-\hat{z}$

$$\bar{H}_0 = S H S^\dagger = \frac{1}{2}\omega_z \overset{\substack{S \text{ commutes} \\ \text{with } \sigma_z}}{\sigma_z} + \frac{1}{2}\omega_p \begin{pmatrix} e^{i\omega_r t/2} & 0 \\ 0 & e^{-i\omega_r t/2} \end{pmatrix}\begin{pmatrix} 0 & c-is \\ c+is & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega_r t/2} & 0 \\ 0 & e^{i\omega_r t/2} \end{pmatrix}$$

$$= \frac{1}{2}\omega_z \sigma_z + \frac{1}{2}\omega_p \sigma_x$$ In frame rotating at ω_r, $\vec{B}_{xy} \to B_x$ Constant in time!

$$\bar{H} = \bar{H}_0 + \bar{H}_1 = \frac{1}{2}\overbrace{(\omega_z - \omega_r)}^{=0 \text{ if } \omega_r = \omega_z}\sigma_z + \frac{1}{2}\omega_p \sigma_x$$

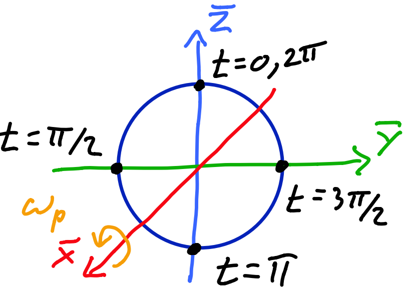
On resonance

$\bar{H} = \frac{1}{2}\omega_p \sigma_x$ $T = e^{-i(\omega_p t/2)\sigma_x}$

$$= \cos\left(\frac{\omega_p t}{2}\right) I - i\sin\left(\frac{\omega_p t}{2}\right)\sigma_x$$

t	T	$\lvert\psi\rangle$
0	I	$\lvert\bar{z}+\rangle$
$\pi/2$	$\frac{1}{\sqrt{2}}(I - i\sigma_x)$	$\lvert\bar{y}-\rangle = \frac{1}{\sqrt{2}}(\lvert\bar{z}+\rangle - i\lvert\bar{z}-\rangle)$
π	$-i\sigma_x$	$-i\lvert\bar{z}-\rangle$
$3\pi/2$	$-\frac{1}{\sqrt{2}}(I + i\sigma_x)$	$\lvert\bar{y}+\rangle = \frac{-1}{\sqrt{2}}(\lvert\bar{z}+\rangle + i\lvert\bar{z}-\rangle)$
2π	$-I$	$-\lvert\bar{z}+\rangle$

units $1/\omega_p$

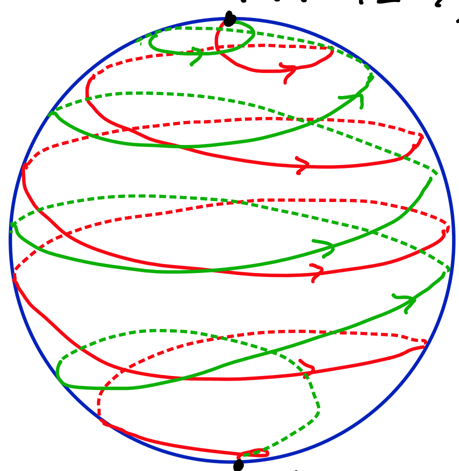


Evolution of $\vec{S}(t)$

one full period $\frac{2\pi}{\omega_p}$

π pulse $t = \frac{\pi}{\omega_p}$

$|\psi_0\rangle = |z+\rangle$, $|\psi_{2\pi}\rangle = -|z+\rangle$



$|\psi_\pi\rangle = -i|z-\rangle$

x-y rotation at frequency ω_r

z rotation at frequency ω_p

Off resonance

Define $\delta = \omega_z - \omega_r \Rightarrow \bar{H} = \frac{1}{2}\begin{pmatrix} \delta & \omega_p \\ \omega_p & -\delta \end{pmatrix}$

Eigenvalues $\lambda = \pm \frac{1}{2}\Omega$, $\Omega = \sqrt{\omega_p^2 + \delta^2}$

Eigenvectors $|\bar{\varphi}_\pm\rangle = A\begin{pmatrix} \delta \pm \Omega \\ -\omega_p \end{pmatrix}$ $\quad A_\pm = \frac{1}{\sqrt{\omega_p^2 + (\delta \pm \Omega)^2}}$

$$T = e^{-it\bar{H}} = e^{-i\Omega t/2}|\bar{\varphi}_+\rangle\langle\bar{\varphi}_+| + e^{i\Omega t/2}|\bar{\varphi}_-\rangle\langle\bar{\varphi}_-|$$

Born Rule

$$P_r([\bar{z}-]_t \,|\, \bar{\psi}_0 = |\bar{z}+\rangle) = |\langle z-|\,T(t)\,|z+\rangle|^2$$

$$= \frac{\omega_p^2}{\omega_p^2 + (\omega_r - \omega_z)^2} \sin^2\left(\frac{\Omega t}{2}\right)$$