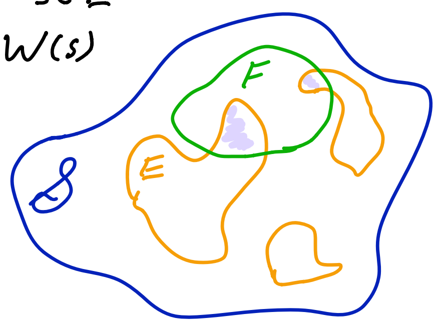
Probability and Statistics

Sample space $\mathcal{S} = \{\text{all possible events}\}$, event $s \in \mathcal{S}$

Random variable $V(s)$ average $\langle V \rangle = \sum_s p_s V(s)$

Example: indicator function $E(s)$, $\langle E \rangle = \sum_{s \in E} p_s = Pr(E)$

Multiple random variables $V(s)$, $W(s)$



Joint probability

$$Pr(\underset{\uparrow \atop \text{event } E}{V=v}, \underset{\uparrow \atop \text{event } F}{W=w}) = \sum_{s \in E \cap F} p_s = \langle EF \rangle$$

Example: Fair die, $\mathcal{S} = \{1 \cdots 6\}$ $V = \text{parity}$ $W = (s-3)^2$

$Pr(v,w)$	$w=0$	$w=1$	$w=4$	$w=9$	$Pr(v) \leftarrow \sum_w Pr(v,w)$
$v=\text{even}$	0	$2/6 = 1/3$	0	$1/6$	$1/2$
$v=\text{odd}$	$1/6$	0	$1/3$	0	$1/2$
$Pr(w)$	$1/6$	$1/3$	$1/3$	$1/6$	$1 \leftarrow \sum_{v,w} Pr(v,w)$

$\uparrow$ $\sum_v Pr(v,w)$

Conditional Probability

$$Pr(A|B) = \underset{\uparrow}{Pr(A,B)} / Pr(B)$$

renormalize $Pr(A,B)$ to subset B is true

Note: $\sum_A Pr(A|B) = 1$

Examples:

$$Pr(v=\text{odd} \,|\, w=4) = Pr(\text{odd},4)/Pr(4) = (1/3)/(1/3) = 1$$

$$Pr(w=4 \,|\, v=\text{odd}) = Pr(\text{odd},4)/Pr(\text{odd}) = (1/3)/(1/2) = 2/3$$

Statistical independence $\Pr(A,B) = \Pr(A)\cdot\Pr(B)$

also check $\neg A, B$ $\quad A, \neg B \quad \neg A, \neg B$

Example: $\Pr(s=1 \text{ or } 6,\ s=\text{even}) = \sum_{s\in\{1,6\}\cap\{2,4,6\}} p_s = p_6 = 1/6$

$\Pr(s=1 \text{ or } 6) = \sum_{s\in\{1,6\}} p_s = 2/6 \qquad \Pr(s=\text{even}) = \sum_{s\in\{2,4,6\}} p_s = 3/6$

$1/6 \stackrel{?}{=} (2/6)\cdot(3/6) = 1/6$ ✓ $\qquad$ also check $s \neq 1$ or 6 and $s=\text{odd}$ ✓

Example: $\Pr(s=2 \text{ or } 6,\ s=\text{even}) = \sum_{s\in\{2,6\}\cap\{2,4,6\}} p_s = p_2 + p_6 = 2/6$

$\Pr(s=2 \text{ or } 6) = \sum_{s\in\{2,6\}} p_s = 2/6 \qquad \Pr(s=\text{even}) = \sum_{s\in\{2,4,6\}} p_s = 3/6$

$2/6 \stackrel{?}{=} (2/6)\cdot(3/6) = 1/6$ ✗ No, not independent

Quantum case

Same, but "single framework rule"

(common event algebra and sample space)

Example: if V, W commute, use $\mathcal{S} = \{\text{common eigenvectors}\}$

Example: $V = v_1 P_1 + v_2 P_2 + v_3 P_3 \qquad W = w_1 P_1 + w_2 Q_2 + w_3 Q_3$

$P_2 Q_2 \neq Q_2 P_2 \qquad P_3 Q_3 \neq Q_3 P_3$

Take $\mathcal{S} = \{P_1, I - P_1\}$ note: $I - P_1 = P_2 + P_3 = Q_2 + Q_3 \Rightarrow (P_2+P_3)(Q_2+Q_3)$
$= (Q_2+Q_3)(P_2+P_3)$!

Can make sense of $\Pr(v_1, w_1), \Pr(\neg v_1, w_1), \Pr(v_1, \neg w_1), \Pr(\neg v_1, \neg w_1)$

Note: $\neg(V = v_1)$ does not imply "$V = v_2$ or v_3" in a framework that also talks about W.

Stochastic Process

probability of sequence of events

Multiple roles of die $\Rightarrow$ sequence $\vec{s} = \{s_0, s_1, s_2 \cdots s_f\}$ in that order

$\vec{s} \in \mathcal{S}_0 \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_f$ (Cartesian product) e.g. $\mathcal{S}_1 = \{\text{even}, \text{odd}\}$ $\mathcal{S}_2 = \{\leq 3, \geq 4\}, \ldots$

$0 \leq Pr(\vec{s}) \leq 1, \quad \sum_{s_0 s_1 \cdots s_f} Pr(s_0 s_1 \cdots s_f) = 1$

Marginal probability $Pr_j(s_j) = \sum_{s_0} \cdots \sum_{s_{j-1}} \sum_{s_{j+1}} \cdots \sum_{s_f} Pr(s_0 \cdots s_j \cdots s_f)$

Note: joint probability $Pr(\vec{s}) \Longrightarrow$ marginals $Pr(s_j)$

reverse $\{Pr_0(s_0), Pr_1(s_1), \cdots, Pr_f(s_f)\} \not\Longrightarrow Pr(\vec{s})$ due to correlations

Independence

$Pr(s_0 s_1 \cdots s_f) = Pr_0(s_0) Pr_1(s_1) \cdots Pr_f(s_f)$

time ordering plays no role, e.g. coin toss $Pr(HHT) = Pr(HTH) = Pr(THH)$

Markov Process

s_{j+1} correlated with s_j but not $s_{j-1} \cdots s_0$

$Pr(s_0, s_1) = Pr(s_1 | s_0) Pr(s_0)$ always true

$Pr(s_0, s_1, s_2) = Pr(s_2 | s_1) Pr(s_0, s_1)$ Sometimes true

$= Pr(s_2 | s_1) Pr(s_1 | s_0) Pr(s_0)$

Define $Pr(s_j | s_{j-1}) \equiv M(s_j, s_{j-1})$ $\longleftarrow$ Markov transition matrix

$Pr(s_j) = M^j \cdot Pr(s_0)$

Note: $\sum_s M(s, s') = 1, \quad M(s,s') \geq 0$ "Stochastic Matrix"