Classical stochastic process: $\mathcal{S}_0\ \mathcal{S}_1 \cdots \mathcal{S}_f$ (sample spaces)

History $\vec{s} = (s_0\, s_1 \cdots s_f) \in \mathcal{S}_0 \times \mathcal{S}_1 \times \cdots \mathcal{S}_f \equiv \tilde{\mathcal{S}}$ Cartesian product, dimensions add

Quantum history Hilbert space: $\mathcal{H}_0 \odot \mathcal{H}_1 \odot \cdots \mathcal{H}_f \equiv \tilde{\mathcal{H}}$

tensor product in time

Basis $\{|\alpha_i\rangle \odot |\beta_j\rangle \odot \cdots |\omega_k\rangle\}$

Generic history $Y = \sum_i \sum_j \cdots \sum_k c_{ij\cdots k} |\alpha_i\rangle \odot |\beta_j\rangle \odot \cdots |\omega_k\rangle \in \tilde{\mathcal{H}}$

Product history $Y^{\vec{\alpha}} = P_0^{\alpha_0} \odot P_1^{\alpha_1} \odot \cdots P_k^{\alpha_k}$

decomposition of identity at time 0

α_0 element of $\{P_0^{\alpha}\}$

$I_0 = \sum_{\alpha_0} P_0^{\alpha_0}$

α_1 element of $I_1 = \sum_{\alpha_1} P_1^{\alpha_1}$

α_k element of $I_k = \sum_{\alpha_k} P_k^{\alpha_k}$

Projector orthogonality $\Rightarrow$ $Y^{\vec{\alpha}} Y^{\vec{\beta}} \equiv (P_0^{\alpha_0} P_0^{\beta_0}) \odot (P_1^{\alpha_1} P_1^{\beta_1}) \odot \cdots (P_k^{\alpha_k} P_k^{\beta_k})$

$= \delta_{\vec{\alpha}\vec{\beta}}\, Y^{\vec{\alpha}}$

$\{Y^{\vec{\alpha}}\}$ serves as product sample space

(mutually exclussive, sums to identity)

Example product sample space: $[z+]_0 \odot [x+]_1$, $[z-]_0 \odot [x+]_1$, $[z+]_0 \odot [x-]_1$, $[z-]_0 \odot [x-]_1$

Check: $\sum_{\vec{\alpha}} Y^{\vec{\alpha}} = I_0 \odot I_1 \equiv \tilde{I}$ $\quad Y^{\vec{\alpha}} Y^{\vec{\beta}} = \delta_{\vec{\alpha}\vec{\beta}}\, Y^{\vec{\alpha}}$ complete set of orthogonal projectors

Example non-product sample space: $[z+]_0 \odot [z+]_1$, $[z-]_0 \odot [x+]_1$, $[z+]_0 \odot [z-]_1$, $[z-]_0 \odot [x-]_1$

Example initial condition $[\psi_0]$, $I_0 = [\psi_0] + (I_0 - [\psi_0])$ decomposition at $t=0$

$Y^{\alpha} = [\psi_0] \odot P_1^{\alpha_1} \odot \cdots P_k^{\alpha_k}$, $Z = (I - [\psi_0]) \odot I_1 \odot \cdots \odot I_k)$ $\Pr(Z) = 0$

Born rule

Single time: $Pr(P_0 | \psi_0) = |P_0|\psi_0\rangle|^2 = \langle\psi_0|P_0|\psi_0\rangle$

time evolution $|\psi_1\rangle = T(t_1, t_0)|\psi_0\rangle$

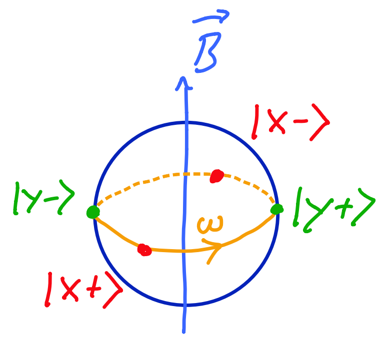
$$Pr(P_1 | \psi_0) = Pr(P_1|\psi_1) = \langle\psi_1|P_1|\psi_1\rangle = \langle\psi_0|T^\dagger P_1 T|\psi_0\rangle$$

Two-time history family: $Y^k = [\psi_0] \odot [\phi_1^k]$ complete basis set at time t_1

$$Z = (I_0 - [\psi_0]) \odot I_1$$

$$Pr(Y^k) = Pr(\phi_1^k | \psi_0) = \langle\psi_0|T^\dagger|\phi_1^k\rangle\langle\phi_1^k|T|\psi_0\rangle = |\langle\phi_1^k|T|\psi_0\rangle|^2$$

Check: $\sum_k Pr(Y^k) + Pr(Z) = 1$ ✓



Example

spin-1/2 in $\vec{B} = B\hat{z}$

$$T = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}$$

Let $|\psi_0\rangle = |x+\rangle_{t=0}$ $\quad \mathcal{S}_0 = \{[x\pm]\}, \mathcal{S}_1 = \{[x\pm]\}$

History family $Y^+ = [x+]\odot[x+]$, $Y^- = [x+]\odot[x-]$, $Z = (I_0 - [x+])\odot I_1$

$$|x+\rangle = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle) \Rightarrow Pr(Y^+) = |\langle x+|T|x+\rangle|^2 = \cos^2(\omega t/2)$$

$$Pr(Y^-) = |\langle x-|T|x+\rangle|^2 = \sin^2(\omega t/2)$$

$$Pr(Z) = \qquad = 0$$

$$\sum = 1$$

- No "measurement"
- No "wavefunction collapse"
- Can extend to future: $[x+]_0 \rightarrow [x-]_1 \rightarrow [x+]_2 \rightarrow \cdots$
- Counterfactuals "what would have occurred if measurement made"