Compatible histories – like compatible properties

$\{Y^\alpha\}$ s.t. $Y^\alpha Y^\beta = Y^\beta Y^\alpha \ \forall \alpha, \beta$

Define logical negation: $\neg Y^\alpha = \tilde{I} - Y^\alpha$

Example: Coin toss $\neg(H,H) = \{(T,H), (H,T), (T,T)\}$

spin-1/2 $\neg [z+] \odot [x+] = [z-] \odot [x+] + [z+] \odot [x-] + [z-] \odot [x-]$

Conjunction: if $YY' = Y'Y$ then $Y \wedge Y' = YY'$

Example: $Y = [z+] \odot I_1 \quad Y' = I_0 \odot [x+] \Rightarrow Y \wedge Y' = [z+] \odot [x+]$

"Start [z+] and end [x+]"

Disjunction: if $YY' = Y'Y$ then $Y \vee Y' = Y + Y' - YY'$

Example: $\left.\begin{array}{l} Y = [z+] \odot ([x+] + [x-]) \\ Y' = ([z+] + [z-]) \odot [x+] \end{array}\right\}$ Same example as above!

$Y + Y' - YY' = (1 + \overset{''}{1} - 1)[z+] \odot [x+] + [z+] \odot [x-] + [z-] \odot [x+]$

"start z+ or end x+"

$= \tilde{I} - [z-] \odot [x-]$

"not 'start z- and end x-'"

<u>Special case</u> Initial state $|\psi_0\rangle$, product history

$Y^{\vec{\alpha}} = [\psi_0] \odot P_1^{\alpha_1} \odot \cdots \odot P_f^{\alpha_f} \in \tilde{\mathcal{H}}$ history Hilbert space

Define <u>chain ket</u> $|\vec{\alpha}\rangle = P_f^{\alpha_f} T_{f,f-1} \cdots T_{2,1} P_1^{\alpha_1} T_{1,0} |\psi_0\rangle \in \mathcal{H}$

ordinary Hilbert space

Generalize Born rule: propose $Pr(\vec{\alpha}) \equiv \langle \vec{\alpha} | \vec{\alpha} \rangle = \| |\vec{\alpha}\rangle \|^2$

"Probability of $P_1^{\alpha_1}$ at t_1 and $P_2^{\alpha_2}$ at t_2 and … $P_f^{\alpha_f}$ at t_f"

Is it correct?

Check two-time history

$Y^{\alpha} = [\psi_0] \odot P_1^{\alpha}$

$|\alpha\rangle = P_1^{\alpha} T_{10} |\psi_0\rangle$

$\langle\alpha|\alpha\rangle = \langle\psi_0| T_{01} P_1^{\alpha} T_{10} |\psi_0\rangle = Pr(P_1^{\alpha}|\psi_0)$ ✓

$\qquad\qquad \langle\psi_1|\; P_1 \;|\psi_1\rangle$

Check "unitary history" (the history that actually happens)

$|\psi_0\rangle \rightarrow |\psi_1\rangle = T_{10}|\psi_0\rangle \rightarrow |\psi_2\rangle = T_{21}|\psi_1\rangle = T_{21}T_{10}|\psi_0\rangle$

$\mathcal{U} = [\psi_0] \odot [\psi_1] \odot [\psi_2] \in \tilde{\mathcal{H}}$ $\quad |\mathcal{U}\rangle = \overbrace{|\psi_2\rangle\langle\psi_2|}^{P_2} T_{21} \overbrace{|\psi_1\rangle\langle\psi_1|}^{P_1} T_{10} |\psi_0\rangle \in \mathcal{H}$

$\langle\psi_2|T_{21} = \langle\psi_1| \qquad \langle\psi_1|T_{10} = \langle\psi_0|$

$\langle\mathcal{U}|\mathcal{U}\rangle = \langle\psi_2|\psi_2\rangle \cdot |\langle\psi_1|\psi_1\rangle|^2 \cdot |\langle\psi_0|\psi_0\rangle|^2 = 1$ ✓

Logical consistency?

Let $\{Y^{\vec{\alpha}}, Z\}$ be a history sample space (mutually compatible and exclussive, sum to identity)

Set $Y^{\vec{\gamma}} = Y^{\vec{\alpha}} + Y^{\vec{\beta}} - \cancelto{0}{Y^{\vec{\alpha}} Y^{\vec{\beta}}}$ "$\vec{\gamma}$ is $\vec{\alpha}$ or $\vec{\beta}$"

Note: $|\vec{\gamma}\rangle = |\vec{\alpha}\rangle + |\vec{\beta}\rangle$ because projectors at each time add

Propose $Pr(\vec{\gamma}) = \langle\vec{\gamma}|\vec{\gamma}\rangle = \langle\vec{\alpha}|\vec{\alpha}\rangle + \langle\vec{\beta}|\vec{\beta}\rangle + \langle\vec{\alpha}|\vec{\beta}\rangle + \langle\vec{\beta}|\vec{\alpha}\rangle$

$\qquad = Pr(\vec{\alpha}) + Pr(\vec{\beta}) + 2\,\mathrm{Re}\,\langle\vec{\alpha}|\vec{\beta}\rangle$

$\qquad = Pr(\vec{\alpha} \text{ or } \vec{\beta}) + 2\,\mathrm{Re}\,\langle\vec{\alpha}|\vec{\beta}\rangle$

$\therefore$ Sufficient condition for consistency $\langle\vec{\alpha}|\vec{\beta}\rangle = 0 \;\; \forall\, \vec{\alpha}, \vec{\beta} \in \tilde{\mathcal{H}}$

Example inconsistent history family (dynamics $T = I$)

$$\left.\begin{aligned} Y^1 &= [z+] \odot [x+] \odot [z+] \\ Y^2 &= [z+] \odot [x+] \odot [z-] \\ Y^3 &= [z+] \odot [x-] \odot [z+] \\ Y^4 &= [z+] \odot [x-] \odot [z-] \\ Z &= [z-] \odot I \odot I \end{aligned}\right\}$$

Complete and Compatible set forms valid sample space

$$|Y^1\rangle = |z+\rangle \underbrace{\langle z+|x+\rangle}_{1/\sqrt{2}} \underbrace{\langle x+|z+\rangle}_{1/\sqrt{2}} = \frac{1}{2}|z+\rangle$$

$Y^1 - Y^4$ all have $Pr(Y^\alpha) \neq 0$

Note: Y^2, Y^4 do not happen but $Pr \neq 0$ due to intermediate amplitudes

$\langle Y^1|Y^3\rangle \neq 0$, $\langle Y^2|Y^4\rangle \neq 0$ Not consistent !

Alternate dynamics ($\vec{B} = B\hat{y}$) $T: |z+\rangle \to |x+\rangle \to |z-\rangle$

$$|Y^1\rangle = |z+\rangle \overbrace{\langle z+|T|x+\rangle}^{0}\langle x+|T|z+\rangle = |0\rangle \quad Pr(Y^1) = 0$$

$$|Y^2\rangle = |z-\rangle\langle z-|T|x+\rangle\langle x+|T|z+\rangle = |z-\rangle \quad Pr(Y^2) = 1$$

Y^2 does happen unitary

$|Y^3\rangle = |0\rangle$ and $|Y^4\rangle = 0$ $\langle Z|Y^1\rangle = 0$ due to initial state

Consistent !!