Example: $Z = (I_0 - [\psi_0]) \odot I_1 \odot \cdots \odot I$ Can it occur?

Chain ket $|Z\rangle = I_f T_{f,f-1} \cdots I_1 T_{1,0} (I_0 - [\psi_0]) |\psi_0\rangle = |0\rangle$

$Pr(Z) = \langle Z | Z \rangle = 0$

Example: Unitary history family

Unitary evolution $|\psi_0\rangle \to |\psi_1\rangle = T_{10}|\psi_0\rangle \longrightarrow |\psi_2\rangle = T_{21}|\psi_1\rangle = T_{20}|\psi_0\rangle$

history of properties $[\psi_0] \odot ([\psi_1] = T_{10}[\psi_0]T_{01}) \odot \quad ([\psi_2] = T_{20}[\psi_0]T_{02})$

$P_0^1 \qquad P_1^1 = T_{10} P_0^1 T_{01} \qquad P_2^1 = T_{20} P_0^1 T_{02}$

$P_0^2 = I_0 - P_0^1 \qquad P_1^2 = I_1 - P_1^1 \qquad P_2^2 = I_2 - P_2^1$

Unitary history family
$$Y^{11\cdots1} = P_0^1 \odot P_1^1 \odot \cdots \odot P_f^1$$
$$Y^{21\cdots1} = P_0^2 \odot P_1^1 \odot \cdots \odot P_f^1$$
$$\vdots \qquad\qquad \vdots$$
$$Y^{22\cdots2} = P_0^2 \odot P_1^2 \odot \cdots \odot P_f^2$$

Check sum: $\sum Y^{\vec{\alpha}} = I_0 \odot I_1 \odot \cdots \odot I_f = \tilde{I}$ ✓

Compatible? $Y^{\vec{\alpha}} Y^{\vec{\beta}} = Y^{\vec{\beta}} Y^{\vec{\alpha}}$

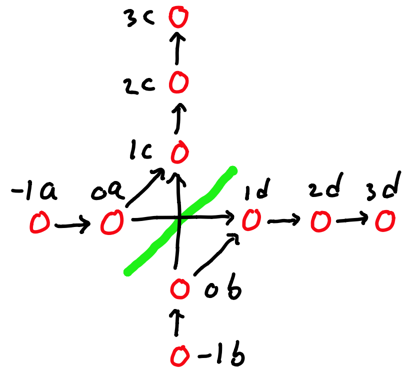
Consistent? Chain ket $|\vec{\alpha}\rangle = ?$

$$|11\cdots1\rangle = (T_{f0} P_0^1 \overbrace{T_{0f})\cdots(T_{20}}^{\cdots} P_0^1 \overbrace{T_{02})\, T_{21}\, (T_{10}}^{I} P_0^1 \overbrace{T_{01})\, T_{10}}^{I} |\psi_0\rangle$$
$$= T_{f0} (P_0^1)^f |\psi_0\rangle = |\psi_f\rangle \qquad Pr(11\cdots1) = \langle 11\cdots1 | 11\cdots1\rangle = 1$$

$|21\cdots1\rangle = T_{f0}(P_0^1)^{f-1} P_0^2 |\psi_0\rangle = |0\rangle \qquad Pr(21\cdots1) = 0 \qquad$ etc.

$\langle \vec{\alpha} | \vec{\beta} \rangle = 0$ unless $\vec{\alpha} = \vec{\beta}$ ✓

Toy beam splitter



Basis $\mathcal{B} = \{|mz\rangle,\ m \in \mathbb{Z},\ z = a, b, c, d\}$

$T = S \qquad S|mz\rangle = |(m+1)z\rangle$

$$S|0a\rangle = \tfrac{1}{\sqrt{2}}(|1c\rangle + |1d\rangle)$$

$$S|0b\rangle = \tfrac{1}{\sqrt{2}}(-|1c\rangle + |1d\rangle)$$

Histories $\quad [0a]_0 \odot \{[1c], [1d]\}_1 \odot \{[2c], [2d]\}_2$

Unitary evolution

$\underline{t=0}\quad |\psi_0\rangle = |0a\rangle$

$\underline{t=1}\quad |\psi_1\rangle = T|\psi_0\rangle = \tfrac{1}{\sqrt{2}}(|1c\rangle + |1d\rangle)$

$\underline{t=2}\quad |\psi_2\rangle = T|\psi_1\rangle = \tfrac{1}{\sqrt{2}}(|2c\rangle + |2d\rangle)$

Chain kets:

$\underline{t=0}\quad |\psi_0\rangle = |0a\rangle$

$\underline{t=1}\quad |(0a\,1c)\rangle = [1c]\,T_{10}|\psi_0\rangle = \tfrac{1}{\sqrt{2}}|1c\rangle$

$|(0a\,1d)\rangle = [1d]\,T_{10}|\psi_0\rangle = \tfrac{1}{\sqrt{2}}|1d\rangle$

$\underline{t=2}\quad |(0a\,1c\,2c)\rangle = [2c]\,T_{21}\,[1c]\,T_{10}|\psi_0\rangle = \tfrac{1}{\sqrt{2}}|2c\rangle$

$|(0a\,1c\,2d)\rangle = [2d]\,T_{21}\,[1c]\,T_{10}|\psi_0\rangle = |0\rangle$

$|(0a\,1d\,2c)\rangle = [2c]\,T_{21}\,[1d]\,T_{10}|\psi_0\rangle = |0\rangle$

$|(0a\,1d\,2d)\rangle = [2d]\,T_{21}\,[1d]\,T_{10}|\psi_0\rangle = \tfrac{1}{\sqrt{2}}|2d\rangle$

Joint probabilities $\quad Pr([1c]_1, [2c]_2 \,|\, [0a]_0) = \langle\alpha|\alpha\rangle = \tfrac{1}{2}$

$$Pr([1d]_1, [2d]_2 \,|\, [0a]_0) = \tfrac{1}{2}$$

Marginal Probabilities $\quad Pr([1c]_1 \,|\, [0a]_0) = \tfrac{1}{2} \qquad Pr([2c]_2 \,|\, [0a]_0) = \tfrac{1}{2}$

$$\Pr(\{1c\}_1 \mid \{2c\}_2) = \Pr(\{1c\}_1, \{2c\}_2 \mid \{0a\}_0) \div \Pr(\{2c\}_2 \mid \{0a\}_0) = \tfrac{1}{2} \div \tfrac{1}{2} = 1 \checkmark$$

interpret the past

$$\Pr(\{1d\}_1 \mid \{2c\}_2) = \Pr(\{1d\}_1, \{2c\}_2 \mid \{0a\}_0) \div \Pr(\{2c\}_2 \mid \{0a\}_0) = 0 \div \tfrac{1}{2} = 0 \checkmark$$