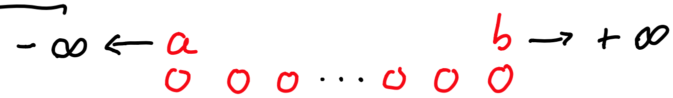
Infinite dimensional Hilbert space

Recall: Hilbert space is complete vector space with inner product

Examples (∞)

1. ℓ^2



(toy model)

Countable infinite basis $\{|m\rangle, m \in \mathbb{Z}\}$

Orthonormality: $\langle m|m'\rangle = \delta_{mm'}$

Completeness: $I = \sum_m |m\rangle\langle m|$ (any $|\psi\rangle = \sum |m\rangle\langle m|\psi\rangle$)

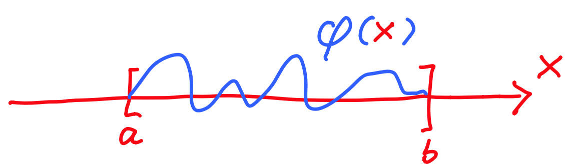
kets: $|\varphi\rangle = \sum c_m |m\rangle$ $\quad |\chi\rangle = \sum d_m |m\rangle$

Components: $c_m = \langle m|\varphi\rangle$

Inner product: $\langle \chi|\varphi\rangle = \langle \chi|\overbrace{\sum_m |m\rangle\langle m|}^{I}|\varphi\rangle = \sum_m d_m^* c_m$

Norm: $\|\varphi\|^2 = \sum_m |c_m|^2 < \infty$ (Hence the name ℓ^2)

2. $L^2[a,b]$



(particle in box)

Uncountable basis $\{|x\rangle, x \in [a,b]\}$

Completeness: $I = \int_a^b dx\, |x\rangle\langle x|$, ket: $|\varphi\rangle$, Component: $\varphi(x) = \langle x|\varphi\rangle$ "Wavefunction"

Orthonormality: $\varphi(x) = \langle x|\int dx' |x'\rangle\langle x'||\varphi\rangle$

$= \int dx' \langle x|x'\rangle \varphi(x') \Rightarrow \langle x|x'\rangle = \delta(x - x')$ "Dirac delta"

Inner product: $\langle \chi | \varphi \rangle = \int_a^b dx\, \chi^*(x)\, \varphi(x)$

Norm: $\| \varphi \|^2 = \int dx\, |\varphi(x)|^2 < \infty$ (Normalizable wave-function)

<u>3. $L^2(\mathbb{R})$</u>

real line $\xrightarrow{\quad}$ x (free particle)

$-\infty \leftarrow a \qquad b \rightarrow \infty$

$\exists$ countable basis

<u>Theorem</u> all seperable ∞-dimensional Hilbert spaces are isomorphic

Example: $L^2[a,b] \equiv \ell^2$

Use Fourier basis (countable $\Rightarrow \ell^2$) to represent function on $[a,b]$

let $\varphi_n(x) = \frac{1}{\sqrt{b-a}}\, e^{i 2\pi n x/(b-a)}$

any $\varphi(x) \in L^2[a,b]$: $\varphi(x) = \sum_{n=-\infty}^{\infty} c_n \varphi_n(x)$

$$c_n = \langle n | \varphi \rangle = \int_a^b dx \frac{1}{\sqrt{b-a}}\, e^{-i 2\pi n x/(b-a)}\, \varphi(x)$$

Can represent any φ: $\| \varphi - \sum_n c_n \varphi_n \| = 0$

<u>Linear operators</u> $|\varphi\rangle \in \mathcal{H} \longrightarrow A|\varphi\rangle$ (maybe not $\in \mathcal{H}$)

Example let $|\varphi\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$, $A\, |\varphi\rangle = \begin{pmatrix} 1 \cdot c_1 \\ 2 \cdot c_2 \\ 3 \cdot c_3 \\ \vdots \end{pmatrix}$

Can have $\| \varphi \|^2 = \sum_n |c_n|^2 < \infty$ but $\| A \varphi \|^2 = \sum_n |n c_n|^2$

Bounded operator $\dfrac{\| A |\varphi\rangle \|}{\| \varphi \|} < M < \infty \quad \forall\, \varphi \in \mathcal{H}$

Example (bounded): $(X\varphi)(x) \equiv x\varphi(x)$ on $L^2[0,1]$

Example (unbounded): $(x\frac{d}{dx}\varphi)(x) \equiv x\varphi'(x)$ on $L^2[0,1]$

same A as above using monomial basis

Eigenvectors & Eigenvalues $A|\varphi\rangle = a|\varphi\rangle$ often lacks solutions in $\mathcal{H}$

Example "pseudo eigenvector"

$A = X, \quad \varphi_a(x) = \delta(x-a) \quad (X\varphi)(x) = x\,\delta(x-a) = a\,\varphi_a(x)$

Note: $\langle \varphi_a | \varphi_b \rangle = \int dx\, \delta(x-a)\,\delta(x-b) = \delta(a-b)$

$A = -i\frac{d}{dx}, \quad \varphi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad (A\varphi_k)(x) = k\,\varphi_k(x)$

Note: $\langle \varphi_k | \varphi_{k'} \rangle = \frac{1}{2\pi}\int dx\, e^{-ikx} e^{ik'x} = \delta(k-k')$

Spectral Decomposition $\{a\}$ can have discrete and continuous parts

$I = \sum_n |n\rangle\langle n| + \int d\nu\, |\nu\rangle\langle\nu|$

$A = \sum_n |n\rangle a_n \langle n| + \int d\nu\, |\nu\rangle a(\nu) \langle \nu|$