# Translations and momentum operator

Recall position basis for $L^2(\mathbb{R})$: $\{|x\rangle\}$, $X|x\rangle = x|x\rangle$

$|\varphi\rangle = \int dx\, \varphi(x)|x\rangle, \qquad \varphi(x) = \langle x|\varphi\rangle$

Translation operator: $U_\alpha: |x\rangle \longrightarrow |x+\alpha\rangle$ Ortho → ortho ⇒ unitary

Translation of state: $\langle x|(U_\alpha|\varphi\rangle) = \langle x|\int dx'\, \varphi(x')\, |x'+\alpha\rangle$

$$= \int dx'\, \varphi(x')\, \delta(x-(x'+\alpha))$$

$$= \varphi(x-\alpha)$$

Translation of operator:

let $A: |\varphi\rangle \longrightarrow |\chi\rangle = A|\varphi\rangle$ Untranslated $A$ does this

$A': U_\alpha|\varphi\rangle \longrightarrow U_\alpha|\chi\rangle = U_\alpha A|\varphi\rangle$ want translated $A$ to do this

$A': U_\alpha|\varphi\rangle \longrightarrow A' U_\alpha|\varphi\rangle$ $A'$ always does this

$$\therefore\ A'U_\alpha = U_\alpha A \Rightarrow A' = U_\alpha A U_\alpha^{-1}$$

Infinitesimal Unitary transformations

$$U(\delta) \approx \underbrace{U(0)}_{I} + \delta \cdot \frac{\partial U}{\partial \alpha}\Big|_{\alpha=0} \equiv I - i\delta T \qquad T \equiv i\frac{\partial U}{\partial \alpha}\Big|_{\alpha=0}$$

$$U^\dagger(\delta) \qquad\qquad = I + i\delta T$$

$$I = U(\delta)U^\dagger(\delta) = (I - i\delta T)(I + i\delta T^\dagger) = I + i\delta\underbrace{\cancel{(T^\dagger - T)}}_{0} + \mathcal{O}(\delta^2)$$

$\Rightarrow T = T^\dagger$ Hermitian

Finite transformations

$$U(\alpha+\delta) = U(\delta)U(\alpha) = (I - i\delta T)U(\alpha) \Rightarrow \partial U/\partial\alpha = -iTU(\alpha)$$

$$\therefore\ U(\alpha) = e^{-i\alpha T}$$

## Translations:

$U(\delta)\varphi(x) = \varphi(x-\delta) \approx \varphi(x) - \delta\varphi'(x) + \cdots$

$I - i\delta T \quad \Rightarrow T = -i\,d/dx$ "generator of translations"

$$U(\alpha) = e^{-\alpha\, d/dx}$$

3-D: $\alpha \to \vec{a}$, $d/dx \to \vec{\nabla}$, $U(\vec{a}) = e^{-\vec{a}\cdot\vec{\nabla}}$

Define "Momentum" $\vec{P} \equiv -i\hbar\vec{\nabla} \qquad U(\vec{a}) = e^{-i\vec{a}\cdot\vec{P}/\hbar}$

Why?

## 1. Commutation relation:

$U(\delta) \approx I - i\delta P/\hbar$: $\quad X \to X' = U(\delta)\, X\, U^\dagger(\delta)$

$$= (I - i\delta P/\hbar)\, X\, (I + i\delta P/\hbar)$$

$$= X + (i\delta/\hbar)(XP - PX) + O(\delta^2)$$

But: $X' = X - \delta I$ is translated by $\delta$

$\therefore\ (i\delta/\hbar)(XP - PX) = -\delta I \quad \Rightarrow [X, P] = i\hbar I$

## 2. Ehrenfest theorem: property $A$ in state $\varphi$

$\langle A\rangle_\varphi(t) = \langle\varphi(t)|A(t)|\varphi(t)\rangle \qquad\qquad i\hbar\frac{d}{dt}|\varphi\rangle = H|\varphi\rangle$

$$\frac{d}{dt}\langle\varphi|A|\varphi\rangle = \left(\frac{d}{dt}\langle\varphi|\right)A|\varphi\rangle + \langle\varphi|\frac{\partial A}{\partial t}|\varphi\rangle + \langle\varphi|A\left(\frac{d}{dt}|\varphi\rangle\right)$$

$$= \langle\varphi|\left(\frac{-1}{i\hbar}H^\dagger\right)A|\varphi\rangle + \langle\varphi|\frac{\partial A}{\partial t}|\varphi\rangle + \langle\varphi|A\left(\frac{1}{i\hbar}H\right)|\varphi\rangle$$

$$= \left(\frac{1}{i\hbar}\right)\langle\varphi|\,[A, H]\,|\varphi\rangle + \langle\varphi|\frac{\partial A}{\partial t}|\varphi\rangle$$

Example $H = \frac{P^2}{2m}$ $A = X$

*Note: $H$ is translation invariant (independent of $X$)*

$$\frac{d}{dt}\langle X \rangle = \frac{1}{i\hbar}\langle [X,H] \rangle$$

$=?$

Recall $[X,P] = i\hbar I$

$$[X,P^2] = XP^2 - P^2X + PXP - PXP$$
$$= P[X,P] + [X,P]P$$
$$= 2i\hbar P$$

$\therefore [X,H] = i\hbar P/m \Rightarrow \frac{d}{dt}\langle X \rangle = P/m$ "velocity"

## 3. Conservation law:

$$\frac{\partial}{\partial t}P = 0 \quad \text{and} \quad [P,H] = 0 \Rightarrow \frac{d}{dt}P = 0$$

$\therefore P$ is conserved quantity arising from translation invariance of $H$

$$H' = e^{-iaP/\hbar}He^{iaP/\hbar} = H$$