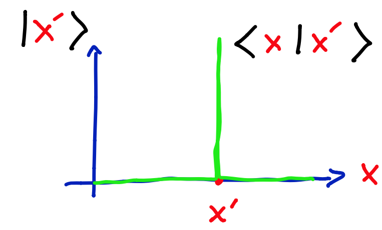
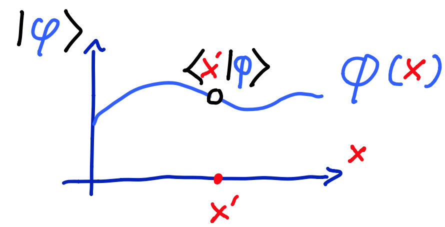
Eigenstates of X $\quad \{|x\rangle,\ x \in \mathbb{R}\}$

$$I = \int dx'\, |x'\rangle\langle x'|$$

$$|\varphi\rangle = I|\varphi\rangle = \int dx'\, |x'\rangle\langle x'|\varphi\rangle$$
$$= \int dx'\, |x'\rangle\, \varphi(x')$$

$$\langle x|x'\rangle = \delta(x-x')$$





Various operators on $|x\rangle$:

$$X|x\rangle = x|x\rangle$$

$$e^{-iaP/\hbar}|x\rangle = |x+a\rangle$$

$$P|x\rangle = ?$$

$$e^{-i\delta P/\hbar}|x\rangle \approx (I - i\delta P/\hbar)|x\rangle = |x\rangle - (i\delta P/\hbar)|x\rangle = |x+\delta\rangle$$

$$P|x\rangle = \lim_{\delta\to 0}\left(\frac{i\hbar}{\delta}\right)\left(|x+\delta\rangle - |x\rangle\right)$$

$$\langle x|P|x'\rangle = \lim_{\delta\to 0}\left(\frac{i\hbar}{\delta}\right)\left(\langle x|x'+\delta\rangle - \langle x|x'\rangle\right) = i\hbar\, \delta'(x-x')$$

$\frac{d}{dx'}$

Operators on $\varphi(x)$:

$$(X\varphi)(x) = \langle x|X|\varphi\rangle = x\langle x|\varphi\rangle = x\varphi(x)$$

$$(P\varphi)(x) = \langle x|P|\varphi\rangle = \int dx' \langle x|P|x'\rangle\langle x'|\varphi\rangle$$

$$= i\hbar\int dx'\, \delta'(x-x')\,\varphi(x') = -i\hbar\,\varphi'(x)$$

assume $\varphi \to 0$ at large x: $\varphi \in L^2(\mathbb{R})$

integrate by parts

Eigenstates of P: $\quad P|p\rangle = p|p\rangle$ defines $|p\rangle$

Variation with x? $\quad \langle x|p\rangle \equiv \chi_p(x)$

Differential equation for $\chi_p(x)$:

$$\langle x|P|p\rangle = p\,\chi_p(x)$$

$$= \langle p|P|x\rangle^* = \Big\{\int dx' \underbrace{\langle p|x'\rangle}_{\chi_p^*(x')} \underbrace{\langle x'|P|x\rangle}_{i\hbar\,\delta'(x'-x)}\Big\}^*$$

$$= -i\hbar\,\chi_p'(x)$$

$$-i\hbar\,\chi_p'(x) = p\,\chi_p(x) \Rightarrow \chi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar}$$

Normalized so that

$$\int dp\, |p\rangle\langle p| = \underline{I}$$

$$\int dx\, \chi_p^*(x)\,\chi_{p'}(x) = \delta(p-p')$$

Position representation: $\quad |\varphi\rangle \longrightarrow \varphi(x) \equiv \langle x|\varphi\rangle$

Momentum representation: $\quad |\varphi\rangle \longrightarrow \tilde{\varphi}(p) \equiv \langle p|\varphi\rangle$

Change of representation:

$$\tilde{\varphi}(p) = \langle p|\varphi\rangle = \langle p|\int dx\, |x\rangle\langle x|\,|\varphi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dx\, e^{-ipx/\hbar}\,\varphi(x)$$

Inverse: $\quad \varphi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int dp\; e^{ipx/\hbar}\,\tilde{\varphi}(p)$

Operators on $\tilde{\varphi}(p)$: $\quad (P\tilde{\varphi})(p) = \langle p|P|\varphi\rangle = p\,\tilde{\varphi}(p)$

$$(X\tilde{\varphi})(p) = \langle p|X\,|\varphi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dx\; x\, e^{-ipx/\hbar}\varphi(x) = i\hbar\frac{\partial}{\partial p}\tilde{\varphi}(p)$$

$I = \int dx\, |x\rangle\langle x|$

Example: Gaussian wave function

$$\varphi(x) = \frac{1}{\sqrt[4]{\pi}\sqrt{\sigma}}\, e^{-x^2/2\sigma^2}$$

Normalized so that $\int dx\, |\varphi(x)|^2 = 1$

Properties: $\langle X \rangle_\varphi = 0 \qquad \langle X^2 \rangle_\varphi = \int dx \;\; x^2\, |\varphi(x)|^2 = \frac{\sigma^2}{2}$

$$\langle P \rangle_\varphi = 0 \qquad \langle P^2 \rangle_\varphi = \int dx\, (-i\hbar)^2 \varphi \frac{d^2}{dx^2} \varphi = \frac{1}{2\sigma^2}\hbar^2$$

$$\tilde{\varphi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{-ipx/\hbar}\, \varphi(x) = \frac{\sqrt{\sigma}}{\sqrt[4]{\pi}\sqrt{\hbar}}\, e^{-p^2\sigma^2/2\hbar^2}$$