Schrödinger equation in momentum representation

$$\langle p | \left[i\hbar \frac{\partial}{\partial t} |\varphi\rangle = \left\{ \frac{P^2}{2m} + V(X) \right\} |\varphi\rangle \right]$$

$$i\hbar \dot{\tilde{\varphi}}(p) = \frac{p^2}{2m} \tilde{\varphi}(p) + \langle p | V | \varphi \rangle$$

?

Let $V(x) = \sum_n V_n x^n$

e.g. $\langle p | V_0 | \varphi \rangle = V_0 \tilde{\varphi}(p)$

$$\langle p | V_1 X | \varphi \rangle = V_1 \frac{1}{\sqrt{2\pi\hbar}} \int dx \, e^{-ipx/\hbar} x \varphi(x) = V_1 \left(i\hbar \frac{\partial}{\partial p} \right) \tilde{\varphi}(p)$$

$$\langle p | V_n X^n | \varphi \rangle = V_n \left(i\hbar \frac{\partial}{\partial p} \right)^n \tilde{\varphi}(p)$$

$\therefore$ $V(X) = V\left(i\hbar \frac{\partial}{\partial p}\right)$ in momentum representation

i.e. Recall $(X\tilde{\varphi})(p) = i\hbar \frac{\partial}{\partial p} \tilde{\varphi}(p) \Rightarrow X = i\hbar \frac{\partial}{\partial p}$

Finally $\langle p | V(X) | \varphi \rangle = V\left(i\hbar \frac{\partial}{\partial p}\right) \tilde{\varphi}(p)$

$$\therefore \; i\hbar \dot{\tilde{\varphi}}(p) = \frac{p^2}{2m} \tilde{\varphi}(p) + V\left(i\hbar \frac{\partial}{\partial p}\right) \tilde{\varphi}(p)$$

Heisenberg "Uncertainty" Relation

Hermitian operators: A, B Commutation $[A,B] = AB - BA \equiv iC$

Anticommutation $\{A,B\} = AB + BA \equiv D$

Mean and variance:

$$\langle A\rangle_\varphi = \langle\varphi|A|\varphi\rangle \qquad \langle A^2\rangle_\varphi = \langle\varphi|A^2|\varphi\rangle$$

$$(\Delta A)^2 \equiv \langle\varphi|\underbrace{(A-\langle A\rangle)^2}_{\tilde{A}^2}|\varphi\rangle = \langle A^2\rangle_\varphi - \langle A\rangle_\varphi^2$$

Schwartz inequality: $\vec{f}, \vec{g}$: $\|\vec{f}\|^2\|\vec{g}\|^2 \underset{= \text{ if } \vec{f} \parallel \vec{g}}{\geq} |\vec{f}\cdot\vec{g}|^2$

Let $\vec{f} = \tilde{A}|\varphi\rangle \qquad \vec{g} = \tilde{B}|\varphi\rangle$

$$(\Delta A)^2 = \|\vec{f}\|^2, \quad (\Delta B)^2 = \|\vec{g}\|^2 \Longrightarrow (\Delta A)^2(\Delta B)^2 \geq |\langle\tilde{A}\tilde{B}\rangle_\varphi|^2$$

$$\tilde{A}\tilde{B} = \frac{\tilde{A}\tilde{B}+\tilde{B}\tilde{A}}{2} + i\,\frac{\tilde{A}\tilde{B}-\tilde{B}\tilde{A}}{2i} = \frac{1}{2}\tilde{D} + \frac{1}{2}i\tilde{C} \leftarrow$$

$\tilde{C} = C$ because $\langle AB\rangle = \langle BA\rangle$

$$\langle\tilde{A}\tilde{B}\rangle = \frac{1}{2}\langle\tilde{D}\rangle + \frac{1}{2}i\langle C\rangle \qquad |\langle\tilde{A}\tilde{B}\rangle|^2 = \frac{1}{4}|\langle\tilde{D}\rangle|^2 + \frac{1}{4}|\langle C\rangle|^2$$

$$\geq \frac{1}{4}|\langle C\rangle|^2$$

$$\therefore\ (\Delta A)(\Delta B) \geq \frac{1}{2}|\langle C\rangle|$$

Example: $A = X \quad B = P \quad C = \hbar I \Rightarrow \Delta X\,\Delta P \geq \frac{1}{2}\hbar$

Equality: $\langle\tilde{D}\rangle = 0$ (Vanishing covariance) and $\tilde{A}|\varphi\rangle \parallel \tilde{B}|\varphi\rangle$

Applications:

1. Hydrogen atom: $H = \frac{p^2}{2m} - \frac{e^2}{R}$

Let $R \sim a_0 \equiv \Delta X$, $p \sim p_0 \equiv \Delta P$ $\quad \Delta X \, \Delta P \sim \hbar \Rightarrow P_0 \sim \frac{\hbar}{a_0}$

$E \sim \frac{\hbar^2}{2m a_0^2} - \frac{e^2}{a_0}$ $\quad$ minimum $E \Rightarrow \frac{\partial E}{\partial a_0} = \frac{-\hbar^2}{m a_0^3} + \frac{e^2}{a_0^2} = 0$

$$\Rightarrow a_0 = \frac{\hbar^2}{me^2} \qquad E_0 = -\frac{me^4}{2\hbar^2}$$

Bohr $\qquad$ Rydberg

2. Optical diffraction limit

Microscope image size $\Delta X \sim h/p = \lambda$ (wavelength)

But: super-resolution microscopy

Temporal Heisenberg Relation

Heisenberg: $\Delta A \; \Delta H \geq \frac{1}{2} \left| \langle [A,H] \rangle \right|$

Ehrenfest: $\frac{d}{dt}\langle A \rangle = \frac{1}{i\hbar} \langle [A,H] \rangle$

$$\Delta A \, \Delta H \geq \frac{1}{2} \hbar \frac{d}{dt} \langle A \rangle$$

Define $\Delta \tau \equiv \Delta A / (d\langle A \rangle / dt)$

$\therefore \; \Delta H \; \Delta \tau \geq \frac{1}{2} \hbar$

<u>Example</u> mass of Z^0 boson, lifetime $\tau \equiv \frac{1}{\Gamma}$

$mc^2 = 91.188$ GeV $\quad \hbar\Gamma = 2.5$ GeV

Note: peak width $\hbar\Gamma \gg$ uncertainty in mc^2