Free Particle Motion

$H = \frac{P^2}{2m} + \cancel{V(X)}^{\,0}$ Energy eigenstate = momentum eigenstate

$H|p\rangle = \frac{p^2}{2m}|p\rangle \equiv E|p\rangle \Rightarrow E = \frac{p^2}{2m}$

Time evolution: $i\hbar \frac{\partial}{\partial t}|\varphi\rangle = H|\varphi\rangle$ $\quad |p\rangle(t) = e^{-iEt/\hbar}|p\rangle(t=0)$

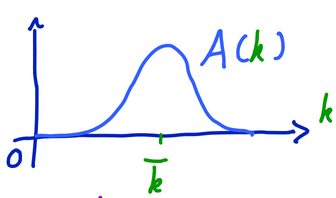
Wave packet: $|\varphi\rangle = \int dp\, \tilde{\varphi}(p)|p\rangle$

$$\varphi(x) = \langle x|\varphi\rangle = \int dp\, \tilde{\varphi}(p)\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dp\, \tilde{\varphi}(p)e^{ipx/\hbar}$$

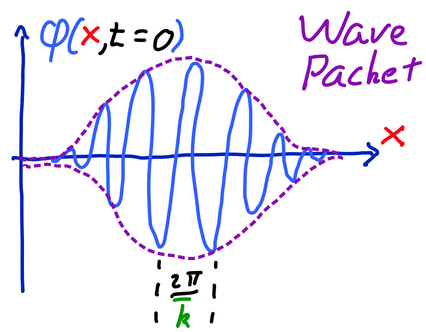
Time evolution: $\varphi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int dp\, \tilde{\varphi}(p,t=0)\, e^{ipx/\hbar - iE(p)t/\hbar}$

New notation: $p = \hbar k$, $E = \hbar\omega$, $\omega = \omega(k)$, $\sqrt{\hbar}\,\tilde{\varphi}(\hbar k) = A(k)$

$$\varphi(x,t) = \frac{1}{\sqrt{2\pi}}\int dk\, A(k)\, e^{i(kx - \omega(k)t)}$$



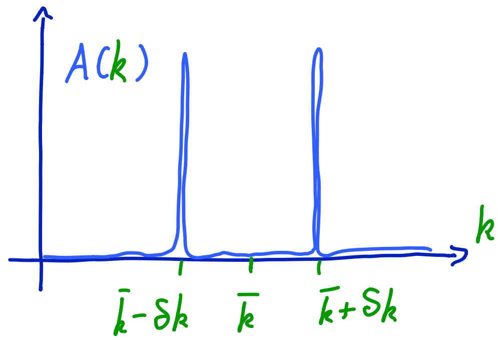
Amplitude



Time evolution?

Example wave packet:

$$A(k) = \tfrac{1}{2}\delta(k-(\bar{k}+\delta k)) + \tfrac{1}{2}\delta(k-(\bar{k}-\delta k))$$



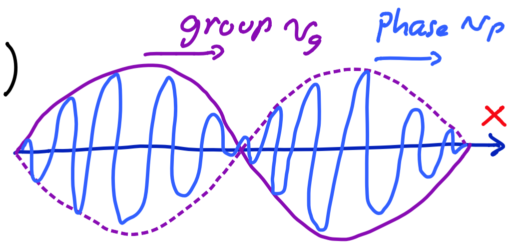
$$\varphi(x,t) = \tfrac{1}{2}e^{i[(\bar{k}+\delta k)x-\underbrace{\omega(\bar{k}+\delta k)}_{(\bar{\omega}+\delta\omega)}t]} + \tfrac{1}{2}e^{i[(\bar{k}-\delta k)x-\underbrace{\omega(\bar{k}-\delta k)}_{(\bar{\omega}-\delta\omega)}t]}$$

$$= e^{i(\bar{k}x-\bar{\omega}t)}\cos(\delta k\,x-\delta\omega t)$$

$x_p = v_p t$ $\qquad$ $x_g = v_g t$

$v_p = \bar{\omega}/\bar{k}$ $\qquad$ $v_g = d\omega/dk$

phase velocity $\qquad$ group velocity



Wavepacket Spreading: Free particle $H = \frac{P^2}{2m}$

Ehrenfest: $\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle[A,H]\rangle$

Recall: $[X,P^2] = 2i\hbar P$

$$\Rightarrow \frac{d}{dt}\langle X\rangle = \frac{1}{m}\langle P\rangle \equiv v_0 \Rightarrow \langle X\rangle = v_0 t + \langle X\rangle_0$$

Also: $[X^2,P^2] = 2i\hbar\{X,P\}$ and $[\{X,P\},P^2] = 4i\hbar P^2$

$$\Rightarrow \frac{d}{dt}\langle X^2\rangle = \frac{1}{m}\langle\{X,P\}\rangle \Rightarrow \frac{d^2}{dt^2}\langle X^2\rangle = \frac{1}{i\hbar m}\langle[\{X,P\},H]\rangle$$

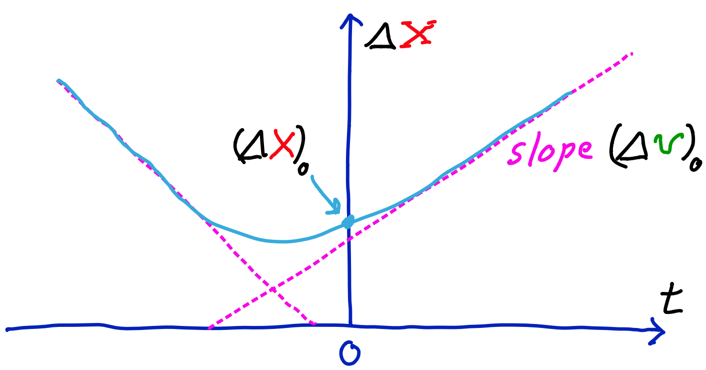
$$= \frac{2\langle P^2\rangle}{m^2}$$

Integrate: $\frac{d}{dt}\langle X^2\rangle = \frac{2\langle p^2\rangle}{m^2}t + \xi_0$ $\xi_0 \equiv \frac{d}{dt}\langle X^2\rangle\Big|_{t=0}$

classical $2v_0x_0$

Integrate again: $\langle X^2\rangle = \frac{\langle p^2\rangle}{m^2}t^2 + \xi_0 t + \langle X^2\rangle_0$

$$\therefore\ (\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2$$
$$= \left(\frac{\langle p^2\rangle_0}{m^2} - v_0^2\right)t^2 + \left(\xi_0 - 2v_0\langle X\rangle_0\right)t + (\Delta X)_0^2$$
$$= (\Delta v)_0^2\, t^2 \qquad + 2\,\Delta(v_0 x_0)\,t + (\Delta X)_0^2$$



Probability current density

Probability density: $\rho(\vec{r}) = Pr([\vec{r}]) = \langle\psi|\vec{r}\rangle\langle\vec{r}|\psi\rangle$

$$= |\psi(\vec{r})|^2$$

Continuity equation: $\frac{\partial}{\partial t}\rho(\vec{r},t) + \vec{\nabla}\cdot\vec{J} = 0$

$$\frac{\partial}{\partial t}|\psi|^2 = \psi^* \frac{\partial}{\partial t}\psi + \psi \frac{\partial}{\partial t}\psi^*$$

$$= \frac{1}{i\hbar}\left(\frac{-\hbar^2}{2m}\right)\left\{\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \underbrace{\;}_{+\nabla\psi^*\cdot\nabla\psi - \nabla\psi\cdot\nabla\psi^*}\right\}$$

$$= \frac{1}{i\hbar}\left(\frac{-\hbar^2}{2m}\right)\vec{\nabla}\cdot\left\{\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right\}$$

Define $\vec{J} \equiv \left(\frac{\hbar}{2im}\right)\left\{\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right\} = Re\left\{\psi^*\left(\frac{\hbar}{im}\right)\vec{\nabla}\psi\right\}$

<u>Example</u>: Plane wave

$\psi(x,t) = e^{i(kx-\omega t)}$, $\quad \frac{\hbar}{im}\frac{\partial}{\partial x}\psi = \frac{\hbar k}{m}\psi \sim v\psi$

$J = Re\, \psi^* v \psi = \rho v$