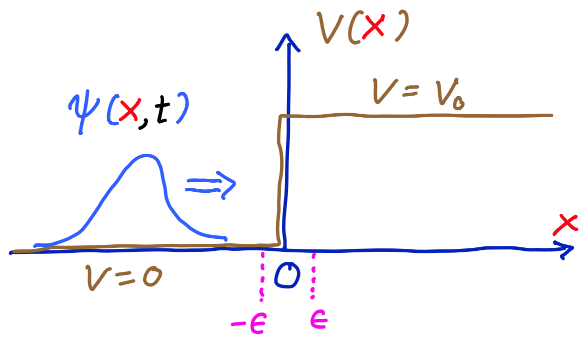
Potential Step

$$H = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H\psi$$



Steady State Solution

$$\psi(x,t) = e^{-iEt/\hbar}\varphi(x) \implies E\varphi(x) = H\varphi(x)$$

Solution $V(x) = 0$

$$\varphi(x) = Ae^{ikx} + Be^{-ikx}, \qquad E = \frac{\hbar^2}{2m}k^2$$

Oscillation if $E > 0$ exponential growth/decay if $E < 0$

Solution $V(x) = V_0$

$$\varphi(x) = Ce^{ik'x} + De^{-ik'x}, \qquad E - V_0 = \frac{\hbar^2}{2m}(k')^2$$

Oscillation if $E - V_0 > 0$ exponential if $E - V_0 < 0$

Discontinuity boundary condition

$$\int_{-\epsilon}^{\epsilon}\left\{\left(\frac{\partial^2}{\partial x^2} + \frac{2m(E - V(x))}{\hbar^2}\right)\varphi(x) = 0\right\}dx = \varphi'(\epsilon) - \varphi'(-\epsilon) + \cancel{\int_{-\epsilon}^{\epsilon}\frac{2m(E-V)\varphi}{\hbar^2}dx} \to 0$$

$\therefore$ $\varphi'(x)$ continuous at $x=0$

φ' finite $\Rightarrow$ $\varphi(x)$ continuous

Example $0 < E < V_0$

$$\varphi(x) = \begin{cases} \overset{\text{incident}}{Ae^{ikx}} + \overset{\text{reflected}}{Be^{-ikx}} & x < 0 \\ \underset{\text{evanescent}}{Ce^{-\kappa x}} + \underset{\text{divergent}}{De^{\kappa x}} & 0 < x \end{cases}$$

$$\kappa \equiv ik' = \sqrt{2m(V_0 - E)/\hbar^2}$$

Continuity of φ: $A + B = C + D$

φ': $ikA - ikB = -\kappa C + \kappa D$

Non-divergent $\Rightarrow D = 0$ $\quad A = 1$ arbitrary

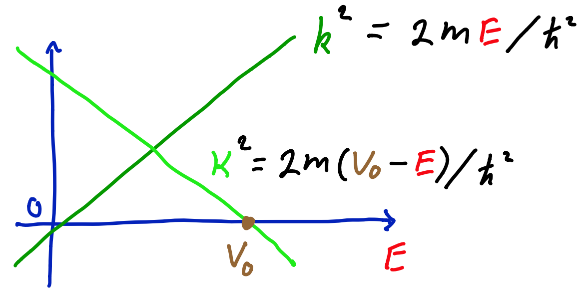
$$\Rightarrow B = -\frac{\kappa + ik}{\kappa - ik} = \frac{1}{B^*}, \quad C = -\frac{2ik}{\kappa - ik}$$

<u>Current density</u> $j = \mathrm{Re}\left\{\frac{\hbar}{im}\varphi^*\varphi'\right\} = \rho v$

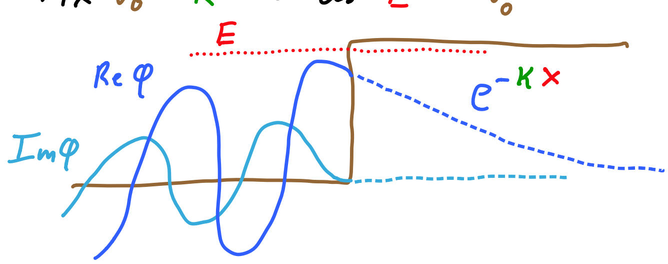
$j_{inc} = |A|^2 \frac{\hbar k}{m}$, $\quad j_{refl} = -|B|^2 \frac{\hbar k}{m}$, $\quad j_{trans} = 0$ because $C^* C \in \mathbb{R}$

$|A|^2 = 1$, $|B|^2 = B^* B = 1 \Rightarrow j_{refl} = -j_{inc}$

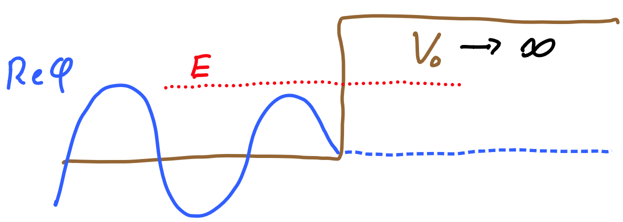
<u>Variation with E</u>



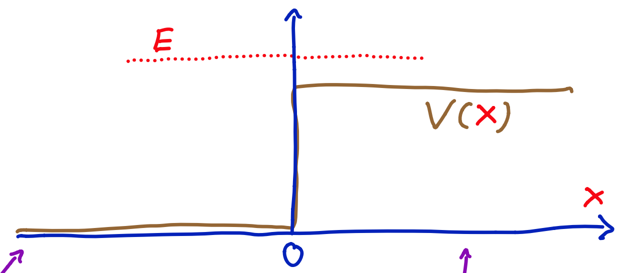
Fix V_0: $\kappa \to 0$ as $E \to V_0^-$



Fix E: as $V_0 \to \infty$,
$$\left.\begin{aligned} &\kappa \to \infty \\ &C \to 0 \\ &B \to -1 \end{aligned}\right\} \quad \varphi(x) = \begin{cases} 0 & x > 0 \\ 2i\sin(kx) & x < 0 \end{cases}$$



Example $0 < V_0 < E$



$\varphi(x) = A e^{ikx} + B e^{-ikx}$ $\qquad$ $\varphi(x) = C e^{ik'x} + D e^{-ik'x}$

$k = \sqrt{2mE/\hbar^2}$ $\qquad$ $k' = \sqrt{2m(E - V_0)/\hbar^2}$

Boundary conditions

Source at $x \to -\infty \Rightarrow A = 1$

No source at $x \to +\infty \Rightarrow D = 0$

φ Continuous at $x = 0$ $\quad A + B = C$

φ' Continuous at $x = 0$ $\quad ik(A - B) = ik'C$

$$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \quad B = \frac{k - k'}{k + k'} \qquad C = \frac{2k}{k + k'}$$

$$j_{inc} = |A|^2 \frac{\hbar k}{m} = v \qquad -j_{refl} = |B|^2 \frac{\hbar k}{m} = \left(\frac{k - k'}{k + k'}\right)^2 \frac{\hbar k}{m} < \frac{\hbar k}{m} = j_{inc}$$

Partial reflection

$$R \equiv |B|^2 / |A|^2 < 1$$

Transmission $\quad T \equiv 1 - R = \dfrac{4kk'}{(k + k')^2} = \dfrac{k'}{k} |C|^2 < |C/A|^2$

because $j = \rho v$ and $v' < v$