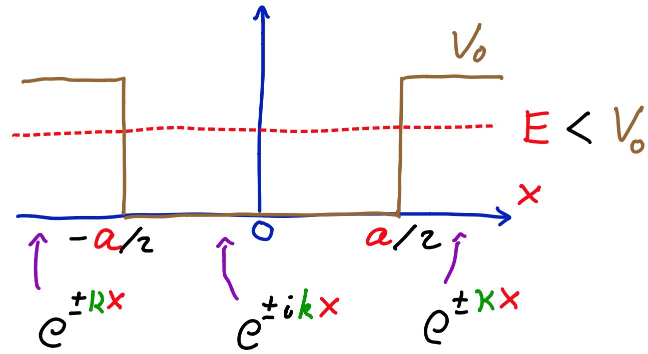
Square Well

$$V(x) = \begin{cases} V_0 & |x| < a/2 \\ 0 & |x| > a/2 \end{cases}$$

$$k = \sqrt{2mE/\hbar^2}$$

$$\kappa = \sqrt{2m(V_0 - E)/\hbar^2}$$



Note: $V(x) = V(-x) \Rightarrow H$ is invariant under Π (parity)

$\Pi: \; x \longrightarrow -x, \quad p \longrightarrow -p \qquad \Pi H \Pi^{-1} = H, \quad [H, \Pi] = 0$

Symmetry group $G = \{I, \Pi\}, \; \Pi^2 = I$ "Z_2"

$\Pi^2 = I \Rightarrow$ eigenvalues of Π are ± 1: $\; \Pi|\pm\rangle = \pm|\pm\rangle$

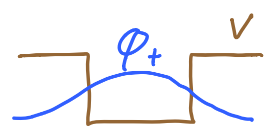
Eigenvectors of H are also eigenvectors of Π:

$\varphi_+(x) = \langle x|+\rangle \qquad \varphi_+(-x) = \langle x|\Pi|+\rangle = \varphi_+(x)$ even

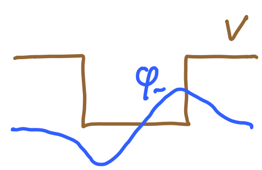
$\varphi_-(x) = \langle x|-\rangle \qquad \varphi_-(-x) = \langle x|\Pi|-\rangle = -\varphi_-(x)$ odd

$\therefore$ Square well solutions:

$$\varphi_+(x) = \begin{cases} A\, e^{-\kappa|x|} & x < -a/2 \\ B \cos(kx) & -a/2 < x < a/2 \\ A\, e^{-\kappa x} & a/2 < x \end{cases}$$



$$\varphi_-(x) = \begin{cases} -A\, e^{-\kappa|x|} & x < -a/2 \\ B \sin(kx) & -a/2 < x < a/2 \\ A\, e^{-\kappa x} & a/2 < x \end{cases}$$



Boundary Conditions

Continuity of $\varphi_+ \Rightarrow B\cos(ka/2) = A e^{-\kappa a/2}$

Continuity of $\varphi_+' \Rightarrow -kB\sin(ka/2) = -\kappa A e^{-\kappa a/2}$

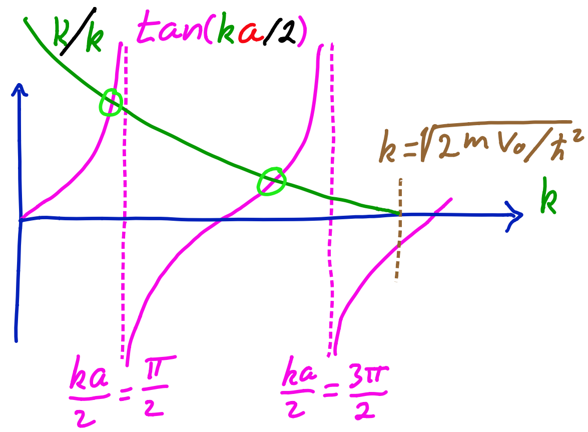
$\div \Rightarrow \tan(ka/2) = \kappa/k$

(For φ_- $\cot\text{an}(ka/2) = -\kappa/k$)

Graphical Solution

(even solutions)

Highest possible bound state $E \leq V_0$



Particle in a box: $V_0 \to \infty$

$\kappa \to \infty$

φ_+ : $\tan(ka/2) = +\infty \Rightarrow k = (2n+1)\pi/a$ Note: $\cos(ka/2)=0$

φ_- : $\cot\text{an}(ka/2) = -\infty \Rightarrow k = (2n)\pi/a$ Note: $\sin(ka/2)=0$

$\uparrow$

BC: $\varphi(x)=0$

Note: all multiples of π/a

$\therefore k_n = n\pi/a \qquad E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2 \qquad n = 1, 2, \cdots$

Opposite limit $V_0 \to 0$: $0 < E < V_0 \quad E \to 0 \quad k, \kappa \to 0$

$\tan(ka/2) = \kappa/k \to ka/2 \qquad \frac{a}{2}k^2 = \kappa \qquad E = \sqrt{V_0 - E} \qquad$ (set $\hbar^2/2m = 1$, $a/2 = 1$)

$0 < E = V_0 - E^2 < V_0$

$\therefore$ Always $\exists$ at least one bound state