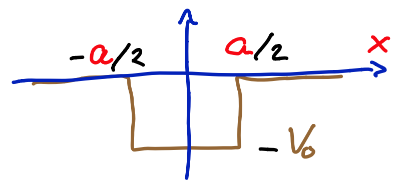
Bound States and Scattering Resonances

Recall $V(x) = \begin{cases} 0 & |x| > a/2 \\ -V_0 & |x| < a/2 \end{cases}$



- Scattering: $E > 0$:

$$\varphi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & (x < -a/2) \\ Ce^{ik'x} + De^{-ik'x} & (-a/2 < x < a/2) \\ Fe^{ikx} + Ge^{-ikx} & (a/2 < x) \end{cases} \qquad \begin{aligned} k &= \sqrt{E} \\ k' &= \sqrt{E - V_0} \end{aligned}$$

Continuity of $\varphi, \varphi' \Rightarrow \quad \begin{pmatrix} A \\ B \end{pmatrix} = M \begin{pmatrix} F \\ G \end{pmatrix}$

Transmission $\Rightarrow \quad G = 0 \qquad T(E) = \left|\frac{F}{A}\right|^2 = \frac{1}{|M_{11}|^2}$

- Bound States: $-V_0 < E < 0$:

$$\varphi(\ ) = \begin{cases} Ae^{-\kappa x} + Be^{\kappa x} & (x < -a/2) \\ \cdots\cdots & \\ Fe^{-\kappa x} + Ge^{\kappa x} & (a/2 < x) \end{cases} \qquad \kappa \equiv -ik = \sqrt{-E}$$

Normalizability $\Rightarrow \quad A = G = 0, \qquad B, F \neq 0$

$\begin{pmatrix} 0 \\ B \end{pmatrix} = M \begin{pmatrix} F \\ 0 \end{pmatrix} \Rightarrow \quad M_{11} = 0 \qquad$ $M_{11}(k = i\kappa)$ has a zero

$T(k = i\kappa)$ has a pole

Bound state → resonance conversion

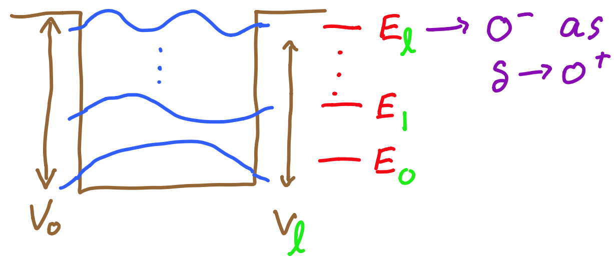
$E \to 0^-$ $\quad K \to 0,\ k' \to \sqrt{V_0}$

$\varphi, \varphi' \Rightarrow$ $\tan(k'a/2) = K/k' = 0 \Rightarrow k' = \frac{2n\pi}{a}$
$\quad\quad\quad \operatorname{cotan}(k'a/2) = -K/k' = 0 \Rightarrow k' = \frac{(2n+1)\pi}{a}$
$\Big\}\ k' = l\frac{\pi}{a}\quad l \in \mathbb{Z}_+$

$\therefore V_l = l^2\left(\frac{\pi}{a}\right)^2$ has $l+1$ bound states (counting $E=0$)

Set $V_0 = V_l + \delta \quad \delta \gtrsim 0$

Highest energy bound state
$E \lesssim 0$



Set $V_0 = V_l + \delta \quad \delta \lesssim 0$

Lowest energy scattering resonance

$$T = \frac{1}{1 + B^2 \sin^2(k'a)} \qquad k' = \sqrt{V_0 + E} = \sqrt{l^2\left(\frac{\pi}{a}\right)^2 + \delta + E}$$
$$= l\left(\frac{\pi}{a}\right) + \mathcal{O}(\delta + E)$$

Resonance at $E = -\delta \gtrsim 0 \to 0^+$ as $\delta \to 0^-$