Scattering

$$V = 0 \qquad V(x) \qquad V = 0$$

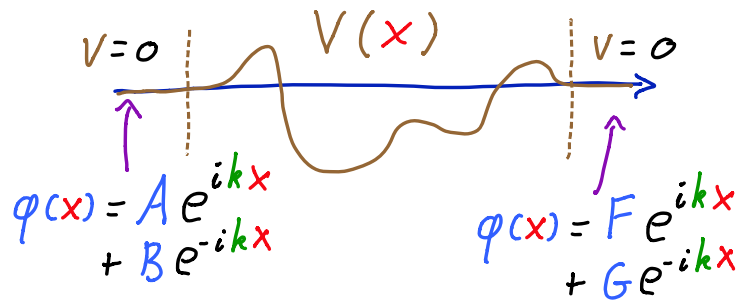$$\varphi(x) = A e^{ikx} + B e^{-ikx} \qquad \varphi(x) = F e^{ikx} + G e^{-ikx}$$

Recall $\begin{pmatrix} A \\ B \end{pmatrix} = M \begin{pmatrix} F \\ G \end{pmatrix}$

Left   Right

$$M = \begin{pmatrix} \gamma & \delta \\ \delta^* & \gamma^* \end{pmatrix}$$

Current Conservation:

$$\det M = |\gamma|^2 - |\delta|^2 = 1$$

Restate $\begin{pmatrix} B \\ F \end{pmatrix} = S \begin{pmatrix} A \\ G \end{pmatrix} \qquad S = \frac{1}{\gamma} \begin{pmatrix} \delta^* & 1 \\ 1 & -\delta \end{pmatrix}$

Out   In

Current Conservation

$$|A|^2 - |B|^2 = |F|^2 - |G|^2 \implies |A|^2 + |G|^2 = |B|^2 + |F|^2$$

R   L   R   L   In   Out

$$\begin{pmatrix} B^* & F^* \end{pmatrix} = \begin{pmatrix} A^* & G^* \end{pmatrix} S^\dagger$$

$$|B|^2 + |F|^2 = \begin{pmatrix} B^* & F^* \end{pmatrix} \begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} A^* & G^* \end{pmatrix} S^\dagger S \begin{pmatrix} B \\ F \end{pmatrix} = |A|^2 + |G|^2$$

$$\therefore \ S^\dagger S = I \qquad S \text{ is unitary}$$

Time reversal $\quad A \leftrightarrow B^* \qquad F \leftrightarrow G^*$

$$\begin{pmatrix} A^* \\ G^* \end{pmatrix} = S \begin{pmatrix} B^* \\ F^* \end{pmatrix} \implies \begin{pmatrix} A \\ G \end{pmatrix} = S^* \begin{pmatrix} B \\ F \end{pmatrix} = S^* S \begin{pmatrix} A \\ G \end{pmatrix} \implies S^T = S \quad \text{symmetric}$$

Tunneling

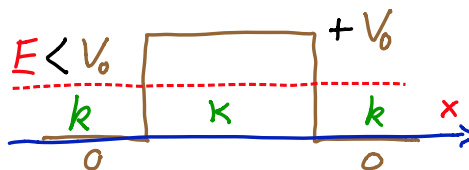Recall square well scattering:

$$k = \sqrt{2mE/\hbar^2}$$
$$k' = \sqrt{2m(E+V_0)/\hbar^2}$$

$$T = \frac{1}{1 + \frac{(k^2 - h'^2)}{2k^2 k'^2} \sin^2(k'a)}$$
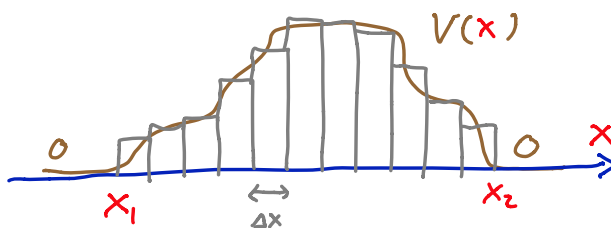
Now consider barrier:

$$-V_0 \longrightarrow + V_0$$
$$k' \longrightarrow \kappa = \sqrt{2m(V_0 - E)/\hbar^2}$$

$$T = \frac{1}{1 + \frac{(k^2 + \kappa^2)}{2k^2 \kappa^2} \sinh^2(\kappa a)} \sim e^{-2\kappa a}$$
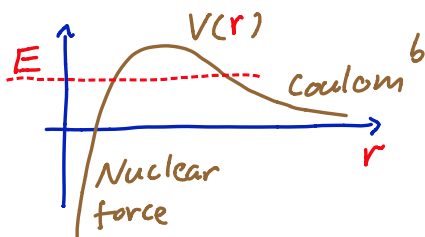
Barrier of arbitrary shape

$$T(x, x+\Delta x) \approx e^{-2\kappa(x)\Delta x}$$

$$T(x_1, x_2) \approx \prod_{i=1}^{N} T(x_i, x_{i+1}) \approx e^{-2\Delta x \sum_{j} \kappa(x_i)} \approx e^{-2\int_{x_1}^{x_2} dx\, \kappa(x)}$$

"WKB Approx."

Applications

1. Nuclear decay

Coulomb

α particle (He^{2+}) is metastable inside nucleus

2. Scanning tunneling microscope

Scan tip $\Rightarrow$ topography or potential

atoms