<u>Periodic Potential</u> $V(x) = V(x+a)$ e.g. electron in crystal

<u>Translational Symmetry</u> $T_a = e^{-a\, d/dx}$, $(T_a V)(x) = V(x-a) = V(x)$

$$H = \frac{p^2}{2m} + V(x) \implies [T_a, H] = 0$$

<u>Conservation law?</u>

Ehrenfest: $\frac{d}{dt}\langle\varphi|T_a|\varphi\rangle = \frac{1}{i\hbar}\langle\varphi|[T_a,H]|\varphi\rangle = 0$ $(\star)$

$\therefore$ $\langle\varphi|T_a|\varphi\rangle$ is conserved But ... is momentum conserved?

No! ... because $[T_\epsilon, H] \neq 0$ for $\epsilon < a$

Some complicated function

Let $\varphi(x, t=0) \equiv \varphi_0 = e^{iqx}$ evolve to $\varphi(x, t>0)$

Note $T_a\varphi(x) = e^{-iqa}\varphi_0(x) \implies \langle\varphi_0|T_a|\varphi_0\rangle = e^{-iqa}$

Time evolution: $(\star) \implies \langle\varphi_t|T_a|\varphi_t\rangle = e^{-iqa}$

Fourier components of $\varphi(x,t) = \int dk\, c_k(t)\, e^{ikx}$ obey $e^{-ika} = e^{-iqa}$

$\therefore$ Only $k = q + \frac{2\pi}{a}n$ allowed $\implies \{q + \frac{2\pi}{a}n\}$ is conserved

<u>Weak conservation law</u> due to discrete translational symmetry (finite a)

Small a limit $\implies$ only q conserved $\implies$ Strong momentum conservation law

Discrete symmetry group $G = \{T_a^n,\ n \in \mathbb{Z}\} \implies \{q + \frac{2\pi}{a}n\}$ conserved

Continuous symmetry group $G = \{T_a,\ a \in \mathbb{R}\} \implies q$ conserved

Bloch Theorem

$[T_a, H] = 0 \Rightarrow$ Simultaneous eigenvectors

Recall $T_a e^{iqx} = e^{-iqa} e^{iqx}$ eigenfunction of T_a but not H

Let $u(x-a) = u(x)$ be periodic $\Rightarrow T_a \big[\underbrace{e^{iqx} u(x)}_{\text{most general eigenfunction of } T_a \text{ (and } H!)}\big] = e^{-iqa} \big[e^{iqx} u(x)\big]$

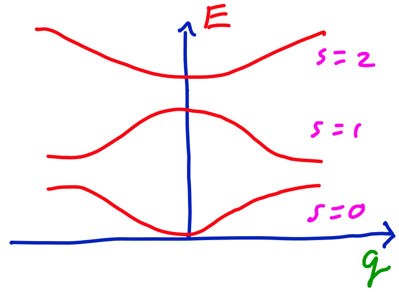
eigenvalue of T_a

$\therefore\; H \varphi_{qs}(x) = E_s(q)\, \varphi_{qs}(x) \qquad \varphi_{qs}(x) = e^{iqx} u_{qs}(x)$

Bloch index

Band index

Dispersion Relation



How to determine u, E?

Schrödinger $\Rightarrow \left\{ \frac{p^2}{2m} + V(x) \right\} e^{iqx} u(x) = E\, e^{iqx} u(x)$

$$\Rightarrow \left\{ \frac{\hbar^2 q^2}{2m} + \frac{\hbar q P}{m} + \frac{P^2}{2m} + V(x) \right\} u(x) = E\, u(x)$$

Eigenvalue equation. Boundary condition $u(x+a) = u(x)$