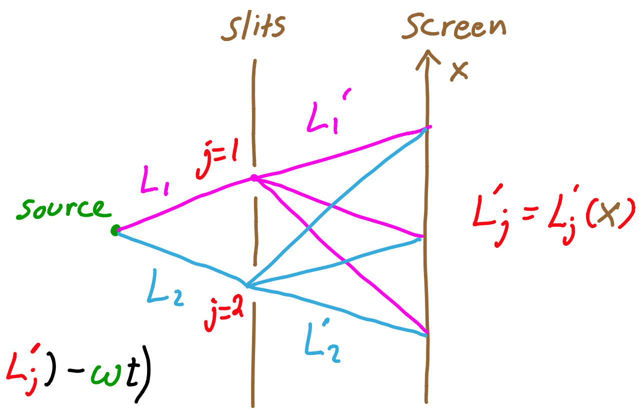
Feynman Path Integrals



Analogy: double slit $j = 1, 2$

Amplitude at slit j: $e^{i(kL_j - \omega t)}$

Amplitude at screen: $\sum_j e^{i(k(L_j + L_j') - \omega t)}$

sum over paths

Unitary evolution

$U(t): |\psi_0\rangle \longrightarrow |\psi_t\rangle = U(t)|\psi_0\rangle$

Say initial state localized at x_0: $\underset{\text{"propagator"}}{U(x,t;x_0)} \equiv \langle x|U(t)|x_0\rangle$

Can solve any initial condition via superposition

$$|\psi_t\rangle = U(t)|\psi_0\rangle = \int dx_0\, U(t)|x_0\rangle\langle x_0|\psi_0\rangle$$

$$\therefore\ \psi(x,t) = \langle x|\psi_t\rangle = \int dx_0\, U(x,t;x_0)\,\psi(x_0, t=0)$$

Massive particle

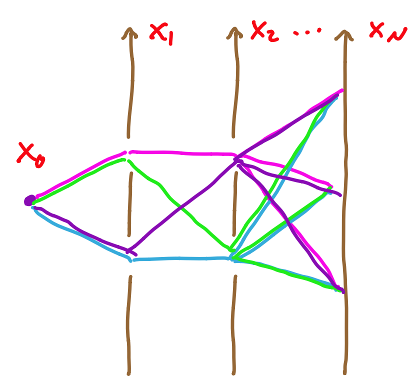
$$H = \frac{p^2}{2m} + V(x) \implies U(t) = e^{-iHt/\hbar}$$

Hard to evaluate if eigenstates unknown

Subdivide time t into slices

N segments size $\epsilon = t/N$:

$$\underset{x_0}{t = 0} \longrightarrow \underset{x_1}{\epsilon} \longrightarrow \underset{x_2}{2\epsilon} \longrightarrow \cdots\ \underset{x_N}{N\epsilon} \equiv t$$



$$t=N\epsilon \longleftarrow (N-1)\epsilon \longleftarrow \cdots \longleftarrow \epsilon \longleftarrow t=0$$

$$U(t) = \underset{\uparrow}{e^{-iH\epsilon/\hbar}}\, e^{-iH\epsilon/\hbar} \cdots \underset{\uparrow}{e^{-iH\epsilon/\hbar}}\, e^{-iH\epsilon/\hbar} \quad \longleftarrow N \text{ factors}$$

$$I_{N-1} = \int dx_{N-1}\, |x_{N-1}\rangle\langle x_{N-1}| \qquad I_1 = \int dx_1\, |x_1\rangle\langle x_1|$$

$$\therefore \langle x | U(t) | x_0 \rangle = \int \prod_{j=1}^{N} dx_j\, \langle x_j | \underbrace{U_\epsilon}_{e^{-iH\epsilon/\hbar}} | x_{j-1} \rangle$$

Evaluation of $\langle x_j | e^{-iH\epsilon/\hbar} | x_{j-1} \rangle$

1. Note: $e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \Longrightarrow$

$$e^{-iH\epsilon/\hbar} = e^{-iV\epsilon/\hbar}\, e^{-iP^2\epsilon/2m\hbar}\, e^{O(\epsilon^2)}$$

2. $\langle x_j | e^{-iH\epsilon/\hbar} | x_{j-1} \rangle = e^{-iV(x_j)\epsilon/\hbar} \langle x_j | e^{-iP^2\epsilon/2m\hbar} | x_{j-1} \rangle$

3. Insert $I = \int dp\, |p\rangle\langle p| \Rightarrow \int \frac{dp}{2\pi\hbar}\, e^{(i/\hbar)[p(x_j - x_{j-1}) - p^2\epsilon/2m]}$

4. Complete the square $\quad ap^2 + bp = (\sqrt{a}\,p + \sqrt{c})^2 - c \qquad c = b^2/4a$

$$\Rightarrow e^{-c} = e^{im(x_j - x_{j-1})^2/2m\epsilon\hbar}$$

5. Gaussian integral $\quad \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{-(i/\hbar)\left(\sqrt{\epsilon/2m}\, p + \sqrt{c}\right)^2} = \sqrt{\frac{m}{2\pi i\hbar\epsilon}}$

$$\therefore U(x_j, \epsilon; x_{j-1}) = \sqrt{\frac{m}{2\pi i\hbar\epsilon}}\, e^{(i/\hbar) L_j \epsilon} \qquad L_j = \frac{1}{2}m\left(\frac{x_j - x_{j-1}}{\epsilon}\right)^2 - V(x_j)$$

"Lagrangian" $L_j = \frac{1}{2} m \dot{x}_j^2 - V(x_j)$

"Velocity" $\dot{x}_j \equiv (x_j - x_{j-1})/\epsilon$

Finally, $U(x,t;x_0) = \int \prod_{j=1}^{N} dx_j \, U(x_j, \epsilon; x_{j-1})$

continuum limit
functional integral $\longrightarrow \int \mathcal{D}[x(t)] \, e^{(i/\hbar)\int_0^t dt' \, L(x,\dot{x})}$

histories

{all trajectories $x(0) = x_0$, $x(t) = x$}

U dominated by path of least action

<u>Example</u> Free particle $V=0$ $\quad x(t) = vt + x_0$

$v = (x - x_0)/t$

$$U(x,t;x_0) = \sqrt{\frac{m}{2\pi i \hbar t}} \, e^{(i/\hbar) m (x-x_0)^2/2t}$$

$$\psi(x,t) = \int dx_0 \, U(x,t;x_0) \, \psi_0(x_0) \longrightarrow \begin{cases} \psi_0(x) & \text{as } t \to 0 \\ \text{Gaussian spreading as } t \uparrow \end{cases}$$