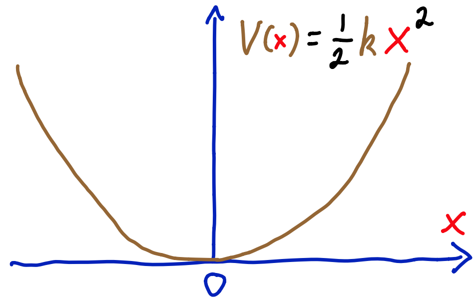
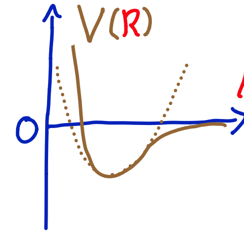
Harmonic Oscillator





Classical: periodic motion

frequency $\omega = \sqrt{k/m}$

$H = \frac{P^2}{2m} + \frac{1}{2}kX^2$

$\qquad \leftarrow \frac{1}{2}m\omega^2X^2$

$[X, P] = i\hbar \qquad H|\varphi\rangle = E|\varphi\rangle$

Expectations:

a) Eigenvalues are real $(H = H^\dagger)$

b) Eigenvalues ≥ 0 $(V \geq 0)$

c) Eigenvectors have definite parity $([\Pi, H] = 0)$

d) Eigenvalues are discrete (bounded motion)

need special accident to have $\varphi(x) \to 0$ as $x \to \pm\infty$

Raising/lowering operators

Scale: $X \to \hat{X} = \sqrt{\frac{m\omega}{\hbar}}\, X \qquad P \to \hat{P} = \frac{1}{\sqrt{m\hbar\omega}} P \qquad [\hat{X}, \hat{P}] = i$

$H \to \hat{H} = \frac{1}{\hbar\omega} H = \frac{1}{2}(\hat{P}^2 + \hat{X}^2)$

Define: $a \equiv \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}) \qquad a \equiv \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P}) \qquad [a, a^\dagger] = 1$

lowering (destruction) raising (creation)

Invert: $\hat{X} = \frac{1}{\sqrt{2}}(a + a^\dagger) \qquad \hat{P} = \frac{1}{\sqrt{2}}(a - a^\dagger)$

Consider: $a^\dagger a = \frac{1}{2}(\hat{X} - i\hat{P})(\hat{X} + i\hat{P}) = \frac{1}{2}(\hat{X}^2 + \hat{P}^2 + i[X, P])$

$\hat{H} = a^\dagger a + \frac{1}{2} \leftarrow$ due to non-commutation

Define $N \equiv a^\dagger a$, $\hat{H} = N + \frac{1}{2}$ $[\hat{H}, N] = 0$

common eigenvectors

possible degeneracy index

$N|\varphi_\nu^i\rangle = \nu|\varphi_\nu^i\rangle \Rightarrow \hat{H}|\varphi_\nu^i\rangle = (\nu + \tfrac{1}{2})|\varphi_\nu^i\rangle \Rightarrow E_\nu = (\nu + \tfrac{1}{2})\hbar\omega$

eigenvector of N eigenvalue

Facts

omit degeneracy index for now

α) $\nu \geq 0$

proof: $\nu = \nu\langle\varphi_\nu|\varphi_\nu\rangle = \langle\varphi_\nu|N|\varphi_\nu\rangle = \langle\varphi|a^\dagger a|\varphi\rangle = |a|\varphi_\nu\rangle|^2 \geq 0$

β) if $\nu = 0$ then $a|\varphi_\nu\rangle = 0$

proof: $0 = \langle\varphi|N|\varphi\rangle = |a|\varphi_\nu\rangle|^2 \Rightarrow a|\varphi_\nu\rangle = 0$

β') if $\nu > 0$ then $N(a|\varphi_\nu\rangle) = (\nu - 1)(a|\varphi_\nu\rangle)$ eigenvalue $\nu - 1$

proof: $[N, a] = [a^\dagger a, a] = a^\dagger[a, a] + [a^\dagger, a]a = -a$

$\Rightarrow N a|\varphi_\nu\rangle = a N|\varphi_\nu\rangle - a|\varphi_\nu\rangle = (\nu - 1) a|\varphi_\nu\rangle$

γ $a^\dagger|\varphi_\nu\rangle \neq 0$

proof: $|a^\dagger|\varphi_\nu\rangle|^2 = \langle\varphi_\nu|a a^\dagger|\varphi_\nu\rangle = \langle\varphi_\nu|(N+1)|\varphi_\nu\rangle > 0$

because $\langle\varphi_\nu|N|\varphi_\nu\rangle \geq 0$

γ' $N(a^\dagger|\varphi_\nu\rangle) = (\nu + 1)(a^\dagger|\varphi_\nu\rangle)$

δ ν is a non-negative integer

proof: Assume $\nu > 0$ not integer $\Rightarrow n < \nu < n+1$

$a^n|\varphi_\nu\rangle$ is eigenvector of N eigenvalue $0 < \nu - n < 1$

Now $a(a^n|\varphi_\nu\rangle)$ is eigenvector of N eigenvalue $-1 < \nu - (n+1) < 0$

Contradiction!

E All levels are nondegenerate

proof: (i) $|\varphi_0\rangle$ is nondegenerate

check: $a|\varphi_0\rangle = \frac{1}{\sqrt{2}}(\hat{X} + \hat{P})|\varphi_0\rangle = 0$

$$\left(x + \frac{d}{dx}\right)\varphi_0(x) = 0 \Rightarrow \varphi_0(x) = c_0\, e^{-x^2/2}$$

unique up to arbitrary complex factor

(ii) induction: if $|\varphi_n\rangle$ nondegenerate then $|\varphi_{n+1}\rangle$ also

check: $a|\varphi^i_{n+1}\rangle = c^i|\varphi_n\rangle$

$$a^\dagger a|\varphi^i_{n+1}\rangle = c^i a^\dagger|\varphi_n\rangle$$

$$|\varphi^i_{n+1}\rangle = \frac{c^i}{n+1}\, a^\dagger|\varphi_n\rangle$$ unique up to complex factor

<u>Eigenstates</u> $\varphi_0(x) = \frac{1}{\sqrt[4]{\pi}}\, e^{-x^2/2}$

By iteration $|\varphi_n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|\varphi_0\rangle$

$$\langle x|\varphi_1\rangle = \sqrt[4]{\frac{4}{\pi}}\; x\, e^{-x^2/2}$$

$$\langle x|\varphi_1\rangle = \sqrt{\frac{1}{4\pi}}\,(2x^2-1)\,e^{-x^2/2}$$

$$\langle x|\varphi_n\rangle \sim H_n(x)\, e^{-x^2/2}$$

Hermite polynomial