WKB (Wentzel Kramers Brillouin) Approximation

Also "semiclassical" see Landau & Lifshitz Quantum

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right)\psi(x) = E\,\psi(x)$$

write $\psi = e^{i\sigma}$

$$\nabla\psi = \left(\frac{i}{\hbar}\nabla\sigma\right)\psi$$

$$\nabla^2\psi = \left(\frac{i}{\hbar}\nabla^2\sigma + \left(\frac{i}{\hbar}\nabla\sigma\right)^2\right)\psi$$

$$\left.\right\} \; -\frac{i\hbar}{2m}\nabla^2\sigma + \frac{1}{2m}(\nabla\sigma)^2 = E - V(x)$$

nonlinear!

Semiclassical limit: small $\hbar$:

$$\sigma = \sigma_0 + (\hbar/i)\sigma_1 + \cdots$$

$\mathcal{O}(\hbar^0)$: $\frac{1}{2m}(\sigma_0')^2 = E - V(x) \Rightarrow \sigma_0' = \pm p(x) \leftarrow$ classical momentum

$$p(x) = \sqrt{2m(E - V(x))}$$

$\sigma_0(x) = \pm\int dx\; p(x)$ classical action

$\psi(x) = e^{\pm (i/\hbar)\int dx\, p(x)}$ oscillates with wavenumber $k(x) = p(x)/\hbar$

Validity: $|\hbar\sigma''| \ll |\sigma'|^2 \Rightarrow d(\underset{\uparrow \atop (2\pi\lambda)}{\hbar/\sigma'})/dx \ll 1 \Rightarrow d\lambda/dx \ll 1/2\pi$

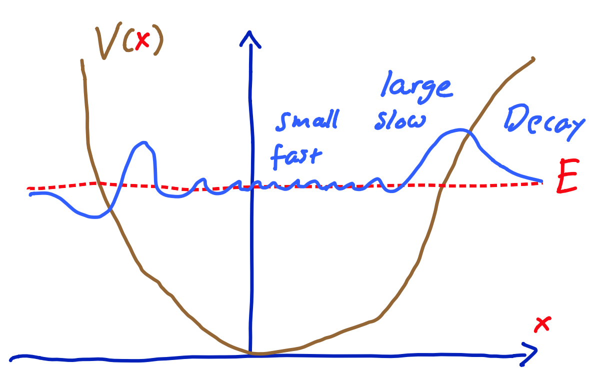
V(x) slowly varying or E large

$\mathcal{O}(\hbar^1)$: $\sigma_0'' + 2\sigma_0'\sigma_1' = 0$

$$\sigma_1' = -\sigma_0''/2\sigma_0' = -p'/2p$$

$$\sigma_1(x) = -\frac{1}{2}\ln p(x)$$

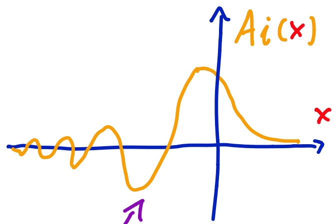
$$\psi(x) = \frac{1}{\sqrt{p}}\left\{ C_1 e^{(i/\hbar)\int p\,dx} + C_2 e^{-(i/\hbar)\int p\,dx}\right\}$$



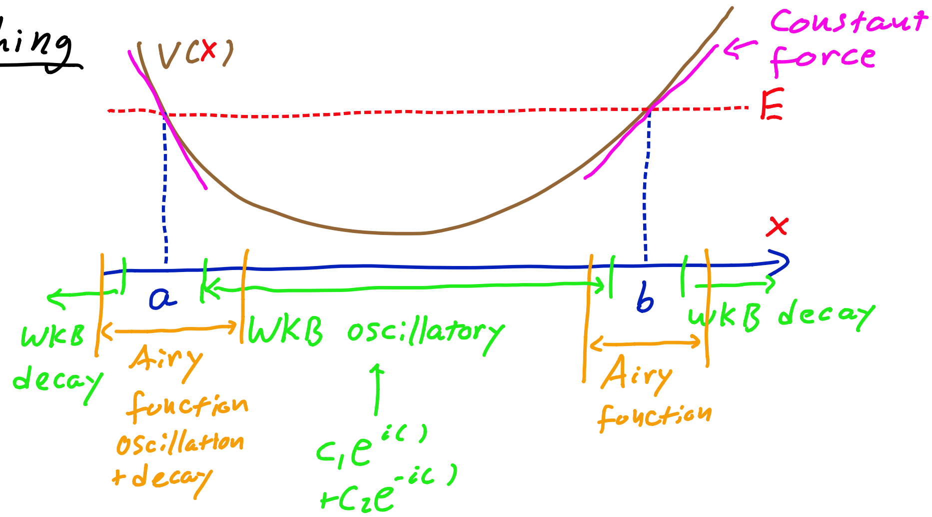
Given E can quickly sketch ψ. But what is E?

Boundary matching

Airy function solution to constant force Schrödinger Eq.



phase $e^{i\pi/4}$



Match at b: $\quad c_1 = Ae^{i\pi/4} \qquad c_2 = Ae^{-i\pi/4}$

$$\psi_{<b}(x) = \frac{c}{\sqrt{p}} \cos\left[\frac{1}{\hbar}\int_b^x p\,dx + \frac{\pi}{4}\right]$$

Match at a: $\quad c_1' = A'e^{-i\pi/4} \qquad c_2' = A'e^{i\pi/4}$

$$\psi_{>a}(x) = \frac{c'}{\sqrt{p}} \cos\left[\frac{1}{\hbar}\int_a^x p\,dx - \frac{\pi}{4}\right]$$

$a < x < b$: $\quad \psi_{>a}(x) = \psi_{<b}(x)$

either $c = c'$ and phases differ by $2\pi m$

or $c = -c'$ and phases differ by $(2m+1)\pi$

Let $c' = (-1)^n c$: $\left[\frac{1}{\hbar}\int_a^x p\,dx - \frac{\pi}{4}\right] - \left[\frac{1}{\hbar}\int_b^x p\,dx + \frac{\pi}{4}\right] = n\pi$

$$\frac{1}{\hbar}\int_a^b p\,dx = \left(n + \tfrac{1}{2}\right)\pi \qquad \oint p\,dx = \left(n + \tfrac{1}{2}\right)h$$

Bohr-Sommerfeld quantization

zero point due to matching

Additional applications

1. Rotational motion $\oint L_z \, d\theta = nh$ no zero point because no Airy matching

2. Energy level spacing:

Compare level n and $n+1$ $\quad \Delta \oint p\, dx = 2\pi\hbar$

$$\approx \Delta E \oint \frac{\partial p}{\partial E} dx$$

$v = \partial E / \partial p$ $\qquad = \Delta E \oint \frac{dx}{v}$

$\therefore \Delta E = \hbar \omega(E)$ $\qquad = \Delta E \cdot T \leftarrow$ period of oscillation

Examples

Harmonic oscillator $\omega = \sqrt{k/m}$ independent of E

$\Rightarrow$ energy levels equally spaced

Square well $\quad \omega(E) \sim v/L \sim \sqrt{E}$

$$\Delta E \sim \sqrt{E} \Rightarrow \frac{dE}{dn} \sim \sqrt{E} \Rightarrow \frac{dE}{\sqrt{E}} \sim n$$

$$\Rightarrow E \sim n^2$$