Harmonic oscillator expectation values

$$H = \frac{P^2}{2m} + V(X) \qquad X = \sqrt{\frac{\hbar}{m\omega}}\,\frac{1}{\sqrt{2}}(a^\dagger + a) \qquad P = \sqrt{\hbar m\omega}\,\frac{i}{\sqrt{2}}(a^\dagger - a)$$

Eigenstates $|\varphi_n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|\varphi_0\rangle \qquad E_n = (n+\frac{1}{2})\hbar\omega$

Mean values $\langle\varphi_n|X|\varphi_n\rangle = \langle\varphi_n|P|\varphi_n\rangle = 0$ in energy eigenstates

Superposition state : $|\psi(t=0)\rangle = \sum_n c_n(0)|\varphi_n\rangle$

evolves to : $\longrightarrow |\psi(t)\rangle = \sum_n c_n(0)\, e^{-iE_n t/\hbar}|\varphi_n\rangle$

Observable A: $= \langle\psi(t)|A|\psi(t)\rangle$

$$= \sum_{mn} c_m^* c_n A_{mn}\, e^{i(E_m - E_n)t/\hbar}$$

$$\frac{E_m - E_n}{\hbar} \equiv \omega_{mn}$$

"Bohr frequency" for $m \leftrightarrow n$ transition

"Selection rule":

ω_{mn} appears in $\langle A\rangle(t)$ only if

$A_{mn} = \langle\varphi_m|A|\varphi_n\rangle \neq 0$ and $c_m^* c_n \neq 0$

Example : $X_{mn} = \sqrt{\frac{\hbar}{2m\omega}}\langle\varphi_m|(a^\dagger + a)|\varphi_n\rangle$

$$= \sqrt{\frac{\hbar}{2m\omega}}\begin{cases}\sqrt{m} & m = n+1\\ \sqrt{n} & m = n-1\end{cases}$$

$\therefore \langle X\rangle(t) \sim e^{i(\pm\hbar\omega)t/\hbar}$

$= e^{\pm i\omega t}$ Same as classical H.O.

<u>Variances</u> $(\Delta X)^2 = \langle \varphi_n | X^2 | \varphi_n \rangle - \langle \varphi_n | X | \varphi_n \rangle^2$

$$X^2 = \frac{\hbar}{2m\omega}(a^\dagger + a)^2 = \frac{\hbar}{2m\omega}(a^{\dagger 2} + a^\dagger a + a a^\dagger + a^2)$$

Note: $\langle \varphi_n | a^{\dagger 2} | \varphi_n \rangle = \langle \varphi_n | a^2 | \varphi_n \rangle = 0$

Note: $a a^\dagger = a^\dagger a + 1 \Rightarrow \langle \varphi_n | (a^\dagger a + a a^\dagger) | \varphi_n \rangle$

$$= \langle \varphi_n | (2 a^\dagger a + 1) | \varphi_n \rangle$$

$$= 2n + 1$$

$(\Delta X)^2 = (n + \frac{1}{2}) \frac{\hbar}{m\omega}$ and $(\Delta P)^2 = (n + \frac{1}{2}) \hbar m \omega$

$\therefore \Delta X \Delta P = (n + \frac{1}{2})\hbar$

<u>Recall</u> $V(X) = \frac{1}{2} m \omega^2 X^2$, $K = \frac{P^2}{2m}$

$\therefore \langle V \rangle_n = \frac{1}{2} E_n = \langle K \rangle$ (Virial Theorem)