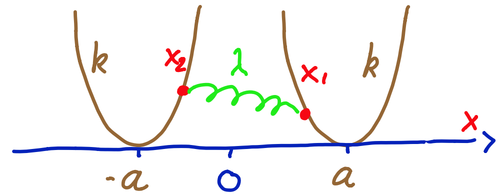
Coupled Oscillators



$$U_0(x_1, x_2) = \frac{1}{2} m\omega^2 \left[(x_1 - a)^2 + (x_2 - a)^2 \right]$$

$$V(x_1, x_2) = \lambda m \omega^2 (x_1 - x_2)^2$$

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + U_0(x_1, x_2) + V(x_1, x_2)$$

Classical solution

$x_G = (x_1 + x_2)/2$ $\quad p_G = p_1 + p_2$ $\quad$ center of mass $\mu_G = 2m$

$x_R = x_1 - x_2$ $\quad p_R = (p_1 - p_2)/2$ $\quad$ relative coord $\mu_R = \frac{m_1 m_2}{m_1 + m_2} = m/2$

$$H = \frac{p_G^2}{2\mu_G} + \frac{1}{2}\mu_G \omega_G^2 x_G^2 \longleftarrow H_G \quad (\omega_G = \omega)$$

$$+ \frac{p_R^2}{2\mu_R} + \frac{1}{2}\mu_R \omega_R^2 \left(x_R - \frac{2a}{1+4\lambda} \right) \longleftarrow H_R \quad (\omega_R = \omega\sqrt{1+4\lambda})$$

displacement due to λ

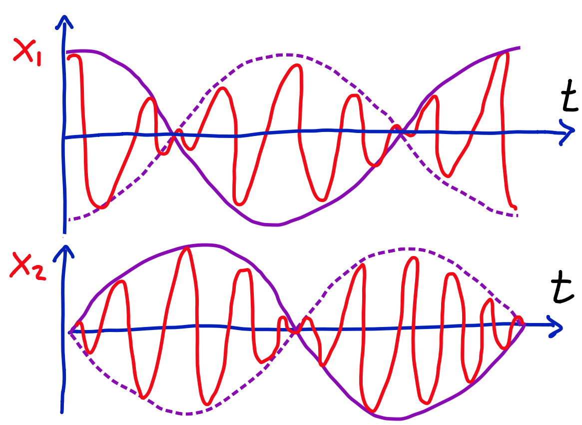
$$+ m\omega^2 a^2 \frac{4\lambda}{1+4\lambda}$$

Normal modes:

$$x_G = x_G^0 \cos(\omega_G t + \theta_G) \qquad x_R = x_R^0 \cos(\omega_R t + \theta_R) + \frac{2a}{1+4\lambda}$$

$$x_{1,2} = x_G \pm \frac{1}{2} x_R$$

Beats:



$$\omega_B = \omega_R - \omega_G$$

$$= \omega\left(\sqrt{1+4\lambda} - 1\right)$$

Quantum Solution

$[X_1, P_1] = [X_2, P_2] = i\hbar \qquad [X_1, P_2] = 0$

$[X_G, P_G] = [X_R, P_R] = i\hbar \qquad [X_G, P_R] = 0$

Hilbert Space: $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}_G \otimes \mathcal{H}_R$

Coupled basis (for $\mathcal{H}_1 \otimes \mathcal{H}_2$)   uncoupled basis (for $\mathcal{H}_G \otimes \mathcal{H}_R$)

Raising & lowering: $a_G = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{\mu_G \omega_G}{\hbar}}\, X_G + i \frac{P_G}{\sqrt{\mu_G \hbar \omega_G}}\right]$

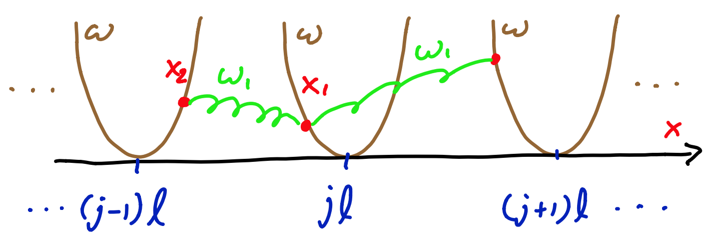
a_R = same, but $X_R' = X_R - \frac{2a}{1+4\lambda}$

$$H = (a_G^\dagger a_G + \tfrac{1}{2})\hbar\omega_G + (a_R^\dagger a_R + \tfrac{1}{2})\hbar\omega_R + \text{cnst.}$$

Spectrum: $|\varphi_{np}\rangle = |\varphi_n^G\rangle|\varphi_p^R\rangle = \frac{1}{\sqrt{n!}}(a_G^\dagger)^n|\varphi_0^G\rangle \frac{1}{\sqrt{p!}}(a_R^\dagger)^p|\varphi_0^R\rangle$

$$E_{np} = (n + \tfrac{1}{2})\hbar\omega_G + (p + \tfrac{1}{2})\hbar\omega_R + \text{cnst.}$$

Harmonic Chain



$$U_0(\cdots x_j \cdots) = \frac{1}{2} m\omega^2 \sum_{j=-\infty}^{\infty} x_j^2$$

$$V(\cdots x_j\; x_{j+1} \cdots) = \frac{1}{2} m\omega_1^2 \sum_j (x_{j+1} - x_j)^2$$

unstretched length ℓ

Force: $F_j = -\frac{\partial(U_0 + V)}{\partial x_j} = -m\omega^2 x_j - m\omega_1^2[(x_j - x_{j+1}) + (x_j + x_{j-1})]$

Solution to $\{F_j = m\ddot{x}_j\}$: Try $x_j = A e^{i(kj\ell - \Omega t)}$ Bloch Thm.!

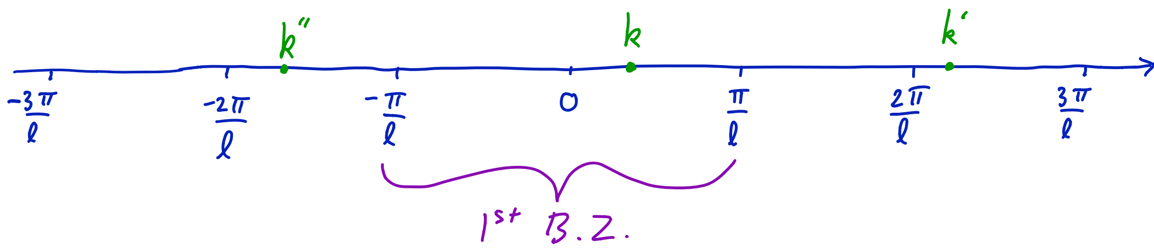
Substitution $\Rightarrow -m\Omega^2 = -m\omega^2 - m\omega_1^2[2 - e^{ik\ell} - e^{-ik\ell}]$

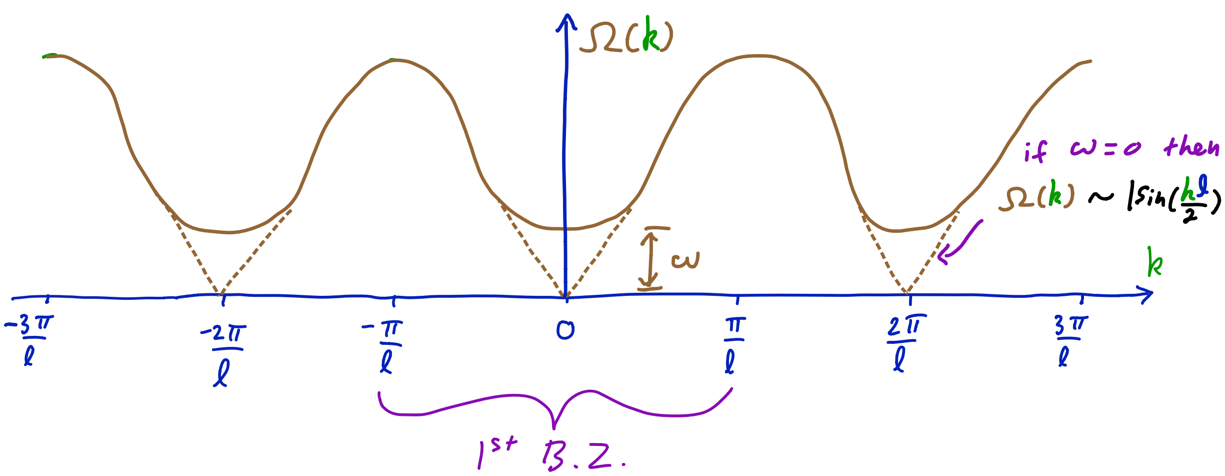
$\Omega = \Omega(k)$ each $k \to 1$ normal mode

First Brillouin Zone

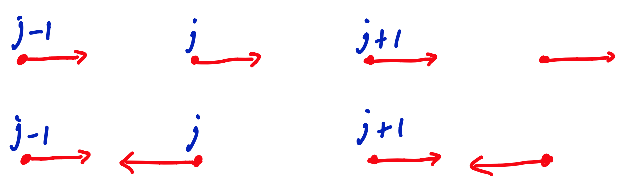
let $k \longrightarrow k' = k + \frac{2\pi}{\ell} n$ $\quad e^{i(k'j\ell - \Omega t)} = e^{i(kj\ell - \Omega t)}\, e^{i\frac{2\pi}{\ell} n j \ell}$

$\therefore\ X_j^{(k')}(t) = X_j^{(k)}(t)$ the same mode: k defined mod $\frac{2\pi}{\ell}$





$k = 0$



$k = 0$: like X_G low Ω

$k = \frac{\pi}{\ell}$: like X_R high Ω