Normal Coordinates

Change of basis $\{x_j(t)\} \longleftrightarrow \{\xi(k,t)\}$

$$\xi(k,t) = \sum_j x_j(t)\, e^{-ikj\ell} \qquad \pi(k,t) = \sum_j p_j(t)\, e^{-ikj\ell}$$

Inverse: $$x_j(t) = \frac{\ell}{2\pi}\int_{-\pi/\ell}^{\pi/\ell} dk\, \xi(k,t)\, e^{ikj\ell} \qquad p_j(t) = \frac{\ell}{2\pi}\int_{-\pi/\ell}^{\pi/\ell} dk\, \pi(k,t)\, e^{ikj\ell}$$

Note: $x_j, p_j \in \mathbb{R} \implies \xi^*(k,t) = \xi(-k,t) \qquad \pi^*(k,t) = \pi(-k,t)$

Identities: $$\sum_{j=-\infty}^{\infty} x_j^2 = \frac{\ell}{2\pi}\int_{-\pi/\ell}^{\pi/\ell} dk\, |\xi(k,t)|^2$$ $\leftarrow$ Parseval relation

$$\sum_{j=-\infty}^{\infty} p_j^2 = \frac{\ell}{2\pi}\int_{-\pi/\ell}^{\pi/\ell} dk\, |\pi(k,t)|^2$$

$$\sum_{j=-\infty}^{\infty} (x_j - x_{j-1})^2 = \frac{\ell}{2\pi}\int_{-\pi/\ell}^{\pi/\ell} dk\, |(1-e^{ik\ell})\,\xi(k,t)|^2$$

$$= \frac{\ell}{2\pi}\int_{-\pi/\ell}^{\pi/\ell} dk\, 4\sin^2\!\left(\frac{k\ell}{2}\right) |\xi(k,t)|^2$$

Hamiltonian: $$H = \sum_{j=-\infty}^{\infty} \left\{ p_j^2/2m + \tfrac{1}{2} m\omega^2 x_j^2 + \tfrac{1}{2} m\omega_1^2 (x_{j+1} - x_j)^2 \right\}$$

$$= \frac{\ell}{2\pi}\int_{-\pi/\ell}^{\pi/\ell} dk \left\{ \frac{1}{2m} |\pi(k,t)|^2 + \frac{1}{2} m\, \Omega^2(k)\, |\xi(k,t)|^2 \right.$$

$$\Omega^2(k) = \omega^2 + 4\omega_1^2 \sin^2\!\left(\frac{k\ell}{2}\right)$$

Let $$\alpha(k,t) \equiv \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\Omega}{\hbar}}\, \xi(k,t) + i\frac{1}{\sqrt{m\hbar\Omega}}\, \pi(k,t)\right]$$

$$\Rightarrow H = \frac{\ell}{2\pi}\int_{-\pi/\ell}^{\pi/\ell} dk\; \frac{1}{2}\hbar\Omega(k) \left\{ \alpha(k,t)\,\alpha^*(k,t) + \alpha(k,t)\,\alpha^*(k,t) \right\}$$

Quantum

$$[X_{j_1}, P_{j_2}] = i\hbar\,\delta_{j_1 j_2}$$

$$\Xi(k) = \sum_j X_j\, e^{-ikjl} = \Xi^\dagger(-k)$$

$$\Pi(k) = \sum_j X_j\, e^{-ikjl} = \Pi^\dagger(-k)$$

$$[\Xi(k), \Pi(k)] = i\hbar \frac{2\pi}{l}\delta(k-k')$$

$$a(k) = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\Omega}{\hbar}}\,\Xi(k) + i\frac{1}{\sqrt{m\hbar\Omega}}\,\Pi(k)\right]$$

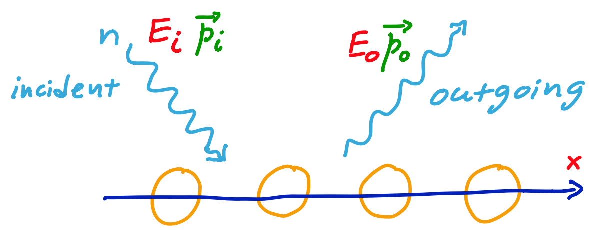
$$H = \frac{l}{2\pi}\int_{-\pi/l}^{\pi/l} dk\ \hbar\Omega(k)\left\{a^\dagger(k)a(k) + \frac{1}{2}\right\}$$

$N(k) \equiv a^\dagger(k)a(k)$ = # of phonons of wavenumber k

Neutron Scattering

$\Delta E = E_o - E_i$

$\Delta \vec{p} = \vec{p}_o - \vec{p}_i$



Neutron interaction $\sim a(\vec{k}) + a^\dagger(\vec{k})$ absorbs or emits phonon $\vec{k}$

$N(\vec{k})$ changes by ± 1

Measurement:

$\vec{k} \equiv \Delta\vec{p}/\hbar$ $\quad\Rightarrow$

$\Omega(\vec{k}) \equiv \Delta E/\hbar$