Orbital angular momentum $\vec{L} = \vec{R} \times \vec{P}$

$L_x = Y P_z - Z P_y$ etc.

$$[L_x, L_y] = [Y P_z - Z P_y, Z P_x - X P_z]$$

Commute

Commute

$$= [Y P_z, Z P_x] + [Z P_y, X P_z]$$

$$= Y [P_z, Z] P_x + X [Z, P_z] P_y$$

$$= -i\hbar Y P_x + i\hbar X P_y = i\hbar L_z$$

Total angular momentum $\vec{L} = \sum_i \vec{L}^{(i)}$ $\quad \vec{J} = \vec{L} + \vec{S} + \cdots$

Define angular momentum by commutation $[J_x, J_y] = i\hbar J_z$

Consider $J^2 \equiv \vec{J}\cdot\vec{J} = J_x^2 + J_y^2 + J_z^2$

$$[J^2, J_x] = [J_x^2 + J_y^2 + J_z^2, J_x] = [J_y^2, J_x] + [J_z^2, J_x]$$

$$= J_y [J_y, J_x] + [J_y, J_x] J_y + J_z [J_z, J_x] + [J_z, J_x] J_z$$

$$= -i\hbar J_y J_z - i\hbar J_z J_y + i\hbar J_z J_y + i\hbar J_y J_z$$

$$= 0$$

$\therefore [J^2, \vec{J}]$ $\exists$ simultaneous eigenstate of J^2 and <u>one of</u> J_x J_y J_z.

Spectrum of J^2 and J_z

Define $J_\pm = J_x \pm i J_y$ $\quad J_- = J_+^\dagger$

Note: $J^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2$ $\qquad [J_z, J_\pm] = \pm\hbar J_\pm$

$[J_+, J_-] = 2\hbar J_z$ $\qquad [J^2, J_\pm] = 0$

Eigenvalues

Note: $\langle \psi | J^2 | \psi \rangle \geq 0 \quad \forall |\psi\rangle \Rightarrow$ eigenvalues of $J^2 \geq 0$

Write: $J^2|j\rangle = j(j+1)\hbar^2|j\rangle \qquad J_z|m\rangle = m\hbar|m\rangle \qquad j, m \in \mathbb{R}$

$j \geq 0$

$\therefore$ Simultaneous eigenstates $|k\,j\,m\rangle$

k ← any additional degeneracy

Facts:

1) $-j \leq m \leq j$ proof:

using $[J_+, J_-]$

$$\begin{aligned} 0 \leq |J_+|jm\rangle|^2 &= \langle jm | J_- J_+ | jm \rangle \\ &= \langle jm | J^2 - J_z^2 - \hbar J_z | jm \rangle \\ &= [\,j(j+1) - m(m+1)\,]\hbar^2 \\ &\Rightarrow m \leq j \end{aligned}$$

also $0 \leq |J_-|jm\rangle|^2$
$\Rightarrow -j \leq m$

2a) If $m = -j$ then $J_-|jm\rangle = 0$ proof: $|J_-|jm\rangle|^2 = [j(j+1) - m(m-1)]$

also if $J_-|jm\rangle = 0$ then $m = -j$

2b) If $m > -j$ then $J_-|jm\rangle$ obeys:

$$J_z(J_-|jm\rangle) = (m-1)(J_-|jm\rangle)$$
$$J^2(J_-|jm\rangle) = j(j+1)(J_-|jm\rangle)$$

proof: $J^2(J_-|jm\rangle) = J_-J^2|jm\rangle = j(j+1)\hbar^2(J_-|jm\rangle)$

$$\begin{aligned} J_z(J_-|jm\rangle) &= J_-J_z|jm\rangle - \hbar(J_-|jm\rangle) \quad \leftarrow \text{using } [J_z, J_\pm] \\ &= m\hbar(J_-|jm\rangle) - \hbar(J_-|jm\rangle) \\ &= (m-1)\hbar(J_-|jm\rangle) \end{aligned}$$

3) Same as 2) but for J_+

4) j is integer or half-integer

Proof: Let $-j \leq m \leq j$. Choose $p \in \mathbb{Z}$ s.t. $-j \leq m + p \leq -j + 1$

Consider $|jm\rangle,\ J_-|jm\rangle,\ \ldots,\ J_-^p|jm\rangle$

Fact 2) $\Rightarrow \exists p$ s.t. $m - p = -j$ (else $\exists |j\,m\rangle$, $m < -j$)

Similarly $\exists q \in \mathbb{Z}$ s.t. $m + q = +j$

Subtract $\Longrightarrow$ $p + q = 2j \in \mathbb{Z}$

$\therefore$ $2j+1$ values of m $\quad -j \le m \le +j$

5) Assume $[A, \vec{J}\,] = 0$ "A is a scalar".

If $\{A, J, J_z\}$ is a CSCO then $A|k\,j\,m\rangle = a_{kj}|k\,j\,m\rangle$ ← indep. of m

proof: Note $A|k\,j\,j\rangle = a_{kj}|k\,j\,j\rangle$ ← $m=j$

$$A|k\,j\,m\rangle = A\,J_-^{\,j-m}|k\,j\,j\rangle = J_-^{\,j-m}A|k\,j\,j\rangle$$
$$= a_{kj}|k\,j\,m\rangle$$

Normalization of eigenstates

Start with $|k\,j\,j\rangle$ normalized.

Define
$$|k\,j\,m{-}1\rangle \equiv \frac{1}{\hbar\sqrt{j(j+1) - m(m-1)}}\,J_-|k\,j\,m\rangle$$
$$|k\,j\,m{+}1\rangle \equiv \frac{1}{\hbar\sqrt{j(j+1) - m(m+1)}}\,J_+|k\,j\,m\rangle$$

Check orthonormality: $\langle k\,j\,m|k'j'm'\rangle = \delta_{kk'}\delta_{jj'}\delta_{mm'}$

e.g. $\langle k\,j\,j{-}1|k\,j\,j{-}1\rangle = \dfrac{1}{2j\hbar^2}\langle k\,j\,j|J_-^\dagger J_-|k\,j\,j\rangle$

$J_+J_- = 2J^2 - 2J_z^2 - \cancel{J_-}J_+ \quad\Longrightarrow\quad = \dfrac{1}{2j\hbar^2}\left(2j(j+1) - 2j^2\right)\hbar^2 = 1$ ✓

Completeness

$$\sum_{kjm}|k\,j\,m\rangle\langle k\,j\,m| = I$$