Matrices representing $\vec{J}$

$\vec{J}$ leaves kj invariant $\Rightarrow$ block diagonal

acts only on m

$$\begin{pmatrix} (j=0) & & \bigcirc \\ & (j=1/2) & \\ \bigcirc & & (j=1) \ \ddots \end{pmatrix}$$

$\underline{j=0}$ only state $m=0$

$J_z |k00\rangle = 0 \Rightarrow J_z = (0)$ 1×1 matrix

$J_\pm |k00\rangle = 0 \Rightarrow J_x = J_y = (0)$ "

$\underline{j=1/2}$ $m = \pm 1/2 \Rightarrow$ 2×2 matrices, basis: $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ $\begin{matrix} \leftarrow m=+\tfrac{1}{2} \\ \leftarrow m=-\tfrac{1}{2} \end{matrix}$

"+" "−"

"↑" "↓"

"0" "1"

$J_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$J_+ = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ flips $|\downarrow\rangle$ to $|\uparrow\rangle$ $J_- = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$ flips $|\uparrow\rangle$ to $|\downarrow\rangle$

$J_x = \frac{1}{2}(J_+ + J_-) = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $J_y = \frac{1}{2i}(J_+ - J_-) = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$

Note: $\vec{J} = \frac{\hbar}{2}\vec{\sigma}$ Pauli spin matrices

$$J^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = j(j+1)\hbar^2 I$$

$$\hbar^2\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \hbar^2\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{4}\hbar^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$\underline{j=1}$ $m = +1, 0, -1 \Rightarrow$ 3×3 matrices basis: $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ $\begin{matrix} \leftarrow m=+1 \\ \leftarrow m=0 \\ \leftarrow m=-1 \end{matrix}$

$J_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$ $J_+ = \hbar\begin{pmatrix}0 & \sqrt{2} & 0\\ 0 & 0 & \sqrt{2}\\ 0 & 0 & 0\end{pmatrix}$ $J_- = \hbar\begin{pmatrix}0 & 0 & 0\\ \sqrt{2} & 0 & 0\\ 0 & \sqrt{2} & 0\end{pmatrix}$

$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}$ $J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}$ $J^2 = 2\hbar^2\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = j(j+1)\hbar^2 I$

etc.

Position representation

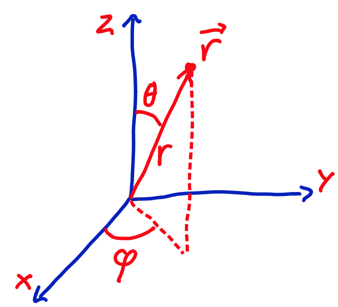
$\vec{R}$ multiplies by $\vec{r}$

$\vec{P} = \frac{\hbar}{i}\vec{\nabla}$

$\vec{L} = \vec{R} \times \vec{P}$ "orbital angular momentum"

$L_x = \frac{\hbar}{i}\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right)$ + cyclic permutations

Spherical polar coordinates



$x = r \sin\theta \cos\varphi$

$y = r \sin\theta \cos\varphi$

$z = r \cos\theta$

$d^3\vec{r} = dx\,dy\,dz = r^2 \sin\theta \, dr\, d\theta\, d\varphi$

$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2 \left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right)$$

$$L_\pm = L_x \pm i L_y = \hbar e^{\pm i\varphi} \left(\pm \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right)$$

$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$ ← By differentiating the coordinate transformation

Eigenfunctions of L^2 + L_z: $\psi_{klm}(r,\theta,\varphi)$

L^2 + L_z indep. of r $\Rightarrow$ $\psi_{klm}(r,\theta,\varphi) = R_{klm}(r) Y_{lm}(\theta,\varphi)$ seperation of variables

$L^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}$ $\quad$ $L_z Y_{lm} = m\hbar Y_{lm}$

Normalization: $\int_0^\infty r^2 dr \, |R(r)|^2 = 1$ and $\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta \, |Y_{lm}|^2 = 1$

Spherical harmonics $Y_{\ell m}(\theta,\varphi)$:

φ - dependence $\quad L_z Y_{\ell m} = m\hbar\, Y_{\ell m} = \frac{\hbar}{i}\frac{\partial}{\partial\varphi} Y_{\ell m}$

$\therefore\ Y_{\ell m}(\theta,\varphi) = F_{\ell m}(\theta)\, e^{im\varphi}$

Note: $Y_{\ell m}(\theta,\varphi+2\pi) = Y_{\ell m}(\theta,\varphi) \implies m \in \mathbb{Z} \implies \ell \in \mathbb{Z}$

θ - dependence $\quad$ Let $m=\ell$. $\quad L_+ Y_{\ell\ell} = 0 \implies \left(\frac{d}{d\theta} - \cot\theta\right) F_{\ell\ell} = 0$

Solution $\quad F_{\ell\ell}(\theta) = c_\ell \sin^\ell\theta$

$$Y_{\ell\ell}(\theta,\varphi) = c_\ell \sin^\ell\theta\, e^{i\ell\varphi}$$

Apply L_- to generate $Y_{\ell m}$ $\quad -\ell \le m \le \ell$

Recall $\quad L_\pm Y_{\ell m} = \hbar\sqrt{\ell(\ell+1) - m(m\pm1)}\; Y_{\ell\, m\pm1}$

Note: $L_\pm$ does not act on $R_{k\ell m}(r) \implies R_{k\ell m} = R_{k\ell}$ indep. of m

Orthonormality $\quad \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta\; Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell'\ell}\, \delta_{m' m}$

Closure $\quad \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta',\varphi') = \frac{1}{\sin\theta}\, \delta(\theta-\theta')\, \delta(\varphi-\varphi')$

Completeness $\quad$ any $f(\theta,\varphi) = \sum_{\ell m} c_{\ell m} Y_{\ell m}(\theta,\varphi)$

with $\quad c_{\ell m} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta\; Y^*_{\ell m}(\theta,\varphi)\, f(\theta,\varphi)$

Special Cases:

$\ell = 0 \Rightarrow m = 0 \qquad Y_{00}(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}$

$\ell = 1 \Rightarrow m = 0, \pm 1 \qquad Y_{1\,\pm 1}(\theta, \varphi) = \mp \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\varphi}$

$$Y_{10}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$\ell = 2 \Rightarrow m = 0, \pm 1, \pm 2 \qquad Y_{2\,\pm 2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{\pm 2i\varphi}$

$$Y_{2\,\pm 1}(\theta, \varphi) = \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{\pm i\varphi}$$

$$Y_{20}(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} \left(3 \cos^2\theta - 1\right)$$

$Y_{\ell 0}(\theta, \varphi) \sim P_\ell(\cos\theta)$

Legendre polynomial