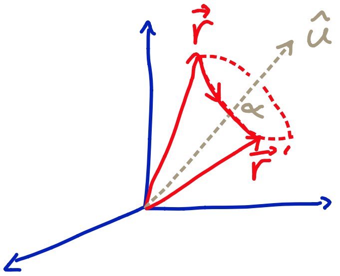
Rotations

$$\vec{\alpha} = \underset{\text{angle}}{\alpha}\,\underset{\text{axis}}{\hat{u}}$$



$R_{\hat u}(\alpha): \vec r \to \vec r\,' = R\,\vec r$ Spatial rotation of vector

Note: $R_{\hat u}(\alpha)\,R_{\hat u}(\alpha') = R_{\hat u}(\alpha')\,R_{\hat u}(\alpha)$ Abelian around common axis

But: $R_{\hat u}(\alpha)\,R_{\hat u'}(\alpha') \neq R_{\hat u'}(\alpha')\,R_{\hat u}(\alpha)$ Non-abelian in general

Rotation of quantum state (spinless)

$R: \vec r \to \vec r\,' = R\,\vec r$

initial state: $\psi(\vec r) = \langle \vec r\,|\,\psi\rangle$

final state: $\psi'(\vec r\,') = \langle \vec r\,'\,|\,\psi'\rangle = \psi(\vec r)$ rigid rotation of function

$\Rightarrow \psi'(R\vec r) = \psi(\vec r) \Rightarrow \psi'(\vec r) = \psi(R^{-1}\vec r)$ like $(T_a\psi)(x) = \psi(x-a)$

dummy variable, rename

Let R be operator in $\mathcal{H}$ s.t. $R: |\psi\rangle \to |\psi'\rangle = R\,|\psi\rangle$

$\therefore \langle \vec r\,|\,(R\,|\psi\rangle) = \langle R^{-1}\vec r\,|\,\psi\rangle$ linear, unitary

Relation to angular momentum (take small $d\alpha$)

$R_{\hat u}(d\alpha): \vec r \to \vec r\,' = R\vec r \approx \vec r + d\alpha\;\hat u \times \vec r$

$\psi'(\vec r) = \psi(R^{-1}\vec r) = \psi(\vec r - d\alpha\;\hat u \times \vec r)$

$$\hat e_z \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

Special case: $\hat u = \hat e_z$ $\psi'(\vec r) = \psi(\vec r - d\alpha\;\hat e_z \times \vec r)$

$= \psi(x + y\,d\alpha,\ y - x\,d\alpha,\ z)$

$$\approx \psi(x,y,z) + d\alpha \left[y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y} \right]$$

$$= \left(1 - \frac{i}{\hbar} d\alpha\, L_z\right)\psi$$

$$\therefore\ R_{\hat{z}}(d\alpha) = 1 - \frac{i}{\hbar} d\alpha\, L_z, \quad R_{\hat{u}}(d\alpha) = 1 - \frac{i}{\hbar} d\alpha\, \hat{u}\cdot\vec{L}$$

$$dR_{\hat{u}}/d\alpha = -\frac{i}{\hbar}\hat{u}\cdot\vec{L} \implies R_{\hat{u}}(\alpha) = e^{-i\alpha\,\hat{u}\cdot\vec{L}/\hbar}$$

↑ Like $T_{\vec{a}} = e^{-i\vec{a}\cdot\vec{P}/\hbar}$

Commutation

$$\underbrace{R_y(-d\alpha')\, R_x(-d\alpha)\, R_y(d\alpha')\, R_x(d\alpha)}_{\text{acts as } R_z(d\alpha\, d\alpha') \text{ on } \vec{r}} = 1 + \left(\frac{i}{\hbar}\right)^2 d\alpha\, d\alpha' [L_x, L_y]$$

$$\implies [L_x, L_y] = i\hbar L_z \quad \checkmark$$

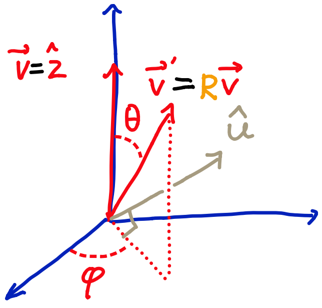
Rotation of spin ½ $\quad \vec{S} = \frac{\hbar}{2}\vec{\sigma}$

Define $R_{\hat{u}}(\theta) = e^{-(i/\hbar)\theta\,\hat{u}\cdot\vec{\sigma}} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2}\,\hat{u}\cdot\vec{\sigma}$

How does R act on spin state $|\chi\rangle$?

Consider $|\chi\rangle = |+\rangle_z \longrightarrow |\chi'\rangle = R|\chi\rangle$

Let $\hat{u} = (\cos(\varphi + \pi/2), \sin(\varphi+\pi/2), 0)$



$$\hat{u}\cdot\vec{\sigma} = -\sin\varphi\,\sigma_x + \cos\varphi\,\sigma_y$$

$$= \frac{1}{2}\left(\sigma_+ e^{-i\varphi} - \sigma_- e^{i\varphi}\right)$$

$$R|+\rangle_z = \cos\frac{\theta}{2}|+\rangle_z + e^{i\varphi}\sin\frac{\theta}{2}|-\rangle_z \quad \leftarrow \text{Recall HW\#3}$$

$$= e^{i\varphi/2}|+\rangle_{\vec{v}'} \qquad R: |\hat{z}+\rangle \longrightarrow |\vec{v}'+\rangle$$

Applications:

1. $H = \frac{P^2}{2m} + V(|\vec{R}|)$ is a scalar

$$[\vec{J}, H] = 0 \Rightarrow \frac{\partial}{\partial t} \langle \vec{J} \rangle = \frac{1}{i\hbar} \langle [\vec{J}, H] \rangle = 0$$

Conservation of angular momentum

2. Let $|kjm\rangle$ be an eigenstate of H eigenvalue E_{kjm}

$$[J_\pm, H] = 0 \quad \Rightarrow \quad H J_+ |kjm\rangle = J_+ H |kjm\rangle = E_{kjm} |kjm\rangle$$

$$\therefore E_{kjm} = E_{kjm+1} \Rightarrow E_{kjm} = E_{kj}$$ indepent of m ⇒ degeneracy $2j+1$

3. Scalar observable A commutes with $\vec{J}$, $J_\pm$ and J^2

⇒ can label eigenstates of A as $|jm\rangle$

⇒ matrix elements $\langle k'j'm' | A | kjm \rangle = \langle k'jj | A | k_{jj} \rangle \delta_{j'j} \delta_{m'm}$

$\uparrow$ indep. of m

Example of Wigner-Eckart theorem