# Rotation of observables

Let $R_{\hat{u}}(\alpha)$ be a rotation, $\{|u_n\rangle\}$ be eigenvectors $A|u_n\rangle = a_n|u_n\rangle$

Define $|u_n'\rangle \equiv R|u_n\rangle$ and $A'$ so that $A'|u_n'\rangle \equiv a_n|u_n'\rangle \quad \forall\, |u_n\rangle$

Then $A'R|u_n\rangle = a_n R|u_n\rangle \Rightarrow R^{-1}A'R = A$

$$A' = RAR^{-1}$$

Infinitesimal $R_{\hat{u}}(d\alpha)$: $\quad A' = \left(1 - \frac{i}{\hbar}d\alpha\, \vec{J}\cdot\hat{u}\right) A \left(1 + \frac{i}{\hbar}d\alpha\, \vec{J}\cdot\hat{u}\right)$

$$= A - \frac{i}{\hbar}d\alpha\,[\vec{J}\cdot\hat{u}, A] + \mathcal{O}(d\alpha^2)$$

Scalar observable: $[\vec{J}, A] = 0 \quad \Rightarrow \quad A' = A$

Examples $\quad P^2 \quad |\vec{R}| \quad H \quad \vec{R}\cdot\vec{P} \quad$ etc.

Vector observable: $\vec{A} = (A_x, A_y, A_z) = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

Can rotate axes

$R_{\hat{x}}(d\alpha)$: $\hat{x} \longrightarrow \hat{x}' = \hat{x}$

$A_x \rightarrow A_x' = \hat{x}'\cdot\vec{A} = A_x$ (no change) $\Rightarrow [J_x, A_x] = 0$

$R_{\hat{y}}(d\alpha)$: $\hat{x} \longrightarrow \hat{x}' = \hat{x} + d\alpha\,\hat{y}\times\hat{x} = \hat{x} - d\alpha\,\hat{z}$

$A_x \rightarrow A_x' = \hat{x}'\cdot\vec{A} = A_x - d\alpha\, A_z \Rightarrow [J_y, A_x] = -i\hbar A_z$

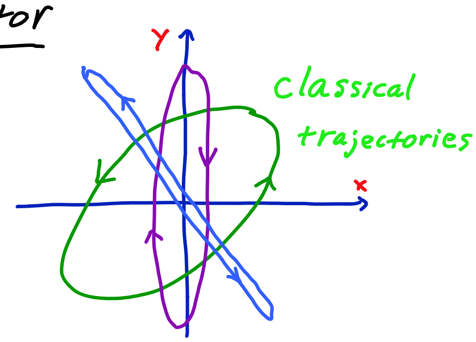
$R_{\hat{z}}(d\alpha)$: $\hat{x} \longrightarrow \hat{x}' = \hat{x} + d\alpha\,\hat{z}\times\hat{x} = \hat{x} + d\alpha\,\hat{y}$

$A_x \rightarrow A_x' = \hat{x}'\cdot\vec{A} = A_x + d\alpha\, A_y \Rightarrow [J_z, A_x] = i\hbar A_y$

$\therefore$ Components of $\vec{A}$ commute with $\vec{J}$ just like components of $\vec{J}$ do

Examples: $\quad \vec{J} \quad \vec{R} \quad \vec{P} \quad \vec{L} = \vec{R}\times\vec{P} \quad$ etc.

# 2-D Symmetric Harmonic Oscillator




$V(x,y) = \frac{1}{2} m\omega^2 (x^2 + y^2) = V(|\vec{r}|)$

$H = \frac{p^2}{2m} + V(x,y) = H_x + H_y$

Raising & lowering $a_x^\dagger \; a_y^\dagger \; a_x \; a_y$

$[a_x, a_y] = 0 \quad [a_x, a_x^\dagger] = 1$ etc.

Number operators $N_x = a_x^\dagger a_x \quad N_y = a_y^\dagger a_y \quad H = (N_x + N_y + 1)\hbar\omega$

Eigenstates $|\varphi_{n_x n_y}\rangle = |n_x\rangle \otimes |n_y\rangle = \frac{1}{\sqrt{n_x!\, n_y!}} (a_x^\dagger)^{n_x} (a_y^\dagger)^{n_y} |\varphi_{00}\rangle$

$E_{n_x n_y} = (n_x + n_y + 1)\hbar\omega \qquad E_n = (n+1)\hbar\omega$

## Spectrum :

$$
\begin{array}{lccccc}
\vdots & \ddots & & \vdots & & \cdot^{\cdot^{\cdot}} \\
n=2 & E_{20} & = & E_{11} & = & E_{02} \\
n=1 & & E_{10} & = & E_{01} & \\
n=0 & & & E_{00} & &
\end{array}
$$

← Degeneracy

$H$ is not a CSCO

Angular momentum $L_z = X P_y - Y P_x = i\hbar (a_x a_y^\dagger - a_x^\dagger a_y)$

$[H, L_z] = 0$ because $H$ is scalar

$\therefore \exists$ common eigenvectors for $H$ and $L_z$ (not $|n_x n_y\rangle$)

## Right & left circular quanta

$a_d = \frac{1}{\sqrt{2}}(a_x - i a_y) \qquad [a_d, a_d^\dagger] = [a_g, a_g^\dagger] = 1 \qquad a_x = \frac{1}{\sqrt{2}}(a_d + a_g)$

$a_g = \frac{1}{\sqrt{2}}(a_x + i a_y) \qquad [a_d, a_g] = 0 \qquad a_y = \frac{1}{\sqrt{2}\, i}(a_g - a_d)$

$a_d$ & $a_g$ contain $a_x$ & $a_y$ $\Rightarrow$ $a_d |n_x\, n_y\rangle = (\ )|n_x-1, n_y\rangle + (\ )|n_x, n_y-1\rangle$

$$H\left(a_{d,g}|n_x,n_y\rangle\right) = E_{n_x+n_y-1}\left(a_{d,g}|n_x,n_y\rangle\right)$$

$$H\left(a^\dagger_{d,g}|n_x,n_y\rangle\right) = E_{n_x+n_y+1}\left(a^\dagger_{d,g}|n_x,n_y\rangle\right)$$

$$E_n = (n+1)\hbar\omega$$

Rewrite $H = \underbrace{(a_d^\dagger a_d + a_g^\dagger a_g + 1)}_{N_d \;+\; N_g \;+\; 1}\hbar\omega$, $\quad L_z = \underbrace{(a_d^\dagger a_d - a_g^\dagger a_g)}_{N_d \;-\; N_g}\hbar$

Eigenstates $\quad |\chi_{n_d n_g}\rangle = \dfrac{1}{\sqrt{n_d!\,n_g!}}(a_d^\dagger)^{n_d}(a_g^\dagger)^{n_g}|\chi_{00}\rangle$

$$H\,|\chi_{n_d n_g}\rangle = (n+1)\hbar\omega\,|\chi_{n_d n_g}\rangle \qquad n \equiv n_d + n_g$$

$$L_z|\chi_{n_d n_g}\rangle = (m+1)\hbar\,|\chi_{n_d n_g}\rangle \qquad m \equiv n_d - n_g$$

Spectrum

| $H$ \ $L_z$ | $m=-2$ | $m=-1$ | $m=0$ | $m=+1$ | $m=+2$ |
|---|---|---|---|---|---|
| $E_2 = 3\hbar\omega$ | $\chi_{02} \sim r^2 e^{-r^2/2} e^{-2i\varphi}$ | | $\chi_{11} \sim (r^2-1)e^{-r^2/2}$ | | $\chi_{02} \sim r^2 e^{-r^2/2} e^{2i\varphi}$ |
| $E_1 = 2\hbar\omega$ | | $\chi_{01} \sim r\, e^{-r^2/2} e^{-i\varphi}$ | | $\chi_{10} \sim r\, e^{-r^2/2} e^{+i\varphi}$ | |
| $E_0 = \hbar\omega$ | | | $\chi_{00} \sim e^{-r^2/2}$ | | |