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33-759 Theoretical Physics Final Exam Monday, Dec. 17, 2007

You may need the following formulas:

Gaussian integral: $\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} = \sqrt{2\pi}\sigma$

For a second order linear differential equation $y'' + p(x)y' + q(x)y$, the Wronskian of any two independent solutions f_1 and f_2 obeys

$$W(x) = f_1 f_2' - f_2 f_1' = W(a) e^{-\int_a^x p(t)dt}$$

The Laplacian in cylindrical coordinates is $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial z^2}$

The Fourier and Laplace transforms are defined as

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \qquad L[f(s)] = \int_0^{\infty} e^{-sx} f(x) dx$$

The Bessel function solves $d^2 J_m(br)/dr^2 + (1/r) dJ_m(br)/dr + (b^2 - m^2/r^2) J_m(br) = 0$

1. Quick answer problems:

(a) Evaluate the Fourier Transform of

$$f(x) = \frac{1}{1+x^2}$$

(b) Evaluate the Laplace Transform of

$$f(t) = \sin(\omega t)$$

(c) Evaluate the integral

$$I = \int_0^\infty \frac{\sin x}{x} dx$$

(d) Obtain the large n asymptotic behavior of

$$I = \int_0^\infty t^n e^{-t} dt$$

(e) Determine $\cos(\pi \mathbf{M}/2)$ where $\mathbf{M}$ is the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

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2. Quantum pillars are nanoscale cylinders inside which electrons can be trapped. Treat the system as a "particle in a box" where the box is a cylinder of length L and radius a, and the electron wavefunction vanishes on the boundaries. Find all solutions of the time-independent Schrödinger equation $-\nabla^2\psi = E\psi$. Your solution should include the eigenfunctions and their eigenvalues. All expressions or quantities introduced must be fully defined.

3. Bessel's differential equation $x^2y'' + xy' + x^2y = 0$ has two linearly independent solutions $J_0(x)$ and $Y_0(x)$ which are known, respectively, as Bessel functions of the first and second kind.

(a) The Bessel function $J_0(x)$ obeys $J_0(0) = 1$. Determine the lowest order correction for small x.

(b) Find the leading behavior of $Y_0(x)$ for small x. Note that your result will contain an arbitrary overall factor.

(c) Find the asymptotic forms of two linearly independent solutions in the limit of large x. Your solutions may be J_0 and Y_0 or any linear combination of the two.