Exact Results on the Antiferromagnetic Three-State Potts Model

Wang, Swendsen, and Kotecký\(^1\) recently presented high-precision numerical data from Monte Carlo simulations of antiferromagnetic three-state Potts models. In two dimensions on a square lattice this model has a zero-temperature critical point related to the six-vertex model at the ice point. We exploit this relation to calculate exact values of the critical exponent \(\gamma/\nu\) and the amplitude of finite-size corrections to the free energy. Our results agree well with the numerical values obtained by Wang, Swendsen, and Kotecký\(^1\).

At \(T = 0\) the antiferromagnetic Potts model finds a ground state in which each spin \(S(r)\equiv \exp[i2\pi \sigma(r)/3]\) \((\sigma = 0,1,2)\) takes on a value different from any of its nearest neighbors. This is the well known three coloring problem, which can be solved exactly by mapping onto a six-vertex model on the dual lattice. Each six-vertex configuration corresponds to three distinct Potts configurations. The partition function of the Potts model is thus 3 times the partition function of an associated six-vertex model.

Care must be taken in relating the boundary conditions of these models. Consider systems of size \(L_x \times L_y\) where, for simplicity, we consider only even values of \(L_x\) and \(L_y\). Imposing periodic boundary conditions on the Potts spins limits the possible six-vertex configurations to those with polarizations of the form \(P_x = 3m\) and \(P_y = 3n\) \((m\) and \(n\) are integers). Polarizations are defined by subtracting the number of down-pointing (or left-pointing) arrows from the number of up-pointing (or right-pointing) arrows and dividing by 2. We will also need boundary conditions, where \(S(r) = S(r + L_x \hat{x}) = \exp(\pm i2\pi/3)\). In this case the allowed polarizations are \(P_x = 3m \pm 1\) and \(P_y = 3n\).

The leading finite-size corrections of a six-vertex model partition function with fixed polarizations are related to those of the Gaussian model partition function with step boundary conditions\(^{3,4}\) so that

\[
Z_{\text{Potts}} \sim 3 \sum_{P_x, P_y} Z_g(\phi_{P_x, P_y}),
\]

with \(Z_g(\phi_{P_x, P_y})\) the partition function of the Gaussian model with \(\phi(r + L_x \hat{x}) = \phi(r) + P_x\) \((k = x, y)\). The Gaussian coupling constant \(K_g\) takes the value of \(2\pi/3\) at the ice point.\(^3\) From this expression we extract the limiting behavior as \(L_x/L_y \to \infty\) (e.g., see Ref. 4). The free energy takes the form

\[
f(L_x, L_y) = f_{\text{bulk}} - \frac{1}{L_x L_y} \ln \left[ \frac{\theta_3(3s) \theta_3(s/3)}{\eta^2(s)} \right],
\]

where \(\theta_3\) is a Jacobi \(\phi\) function of the third kind and \(\eta\) is the Dedekind \(\eta\) function (see Ref. 4 for a definition of \(\theta_3\) and \(\eta\)). \(s = L_y/L_x\) is the aspect ratio of the lattice. The calculations of Wang, Swendsen, and Kotecký\(^1\) were carried out on lattices with \(s = 1\), for which we predict finite-size corrections of the form \(A/L_x^2\) with \(A = \ln 2.93577965\ldots\). This value of \(A\) fits the numerical data very well.

Wang, Swendsen, and Kotecký\(^1\) also obtained a value of the critical exponent \(\gamma/\nu = 1.666(2)\) from the size dependence of the staggered magnetization. We find it is exactly \(\frac{1}{2}\). The idea is to force an interface into the Potts spin system by employing step boundary conditions. Repeating the previous calculation with the new boundary we find an excess free energy per unit length for the interface, \(\zeta(s)\). In the infinite-cylinder limit, \(s \to \infty\), the interfacial free energy becomes

\[
\zeta(s = \infty) = 2\pi x_c/L_x.
\]

We identify \(x_c = 1/3\) as the staggered magnetization critical exponent. This value of \(x_c\) can also be obtained from the work of den Nijs, Nightingale, and Schick.\(^5\) We then obtain the critical exponent for staggered susceptibility from the elementary finite-size-scaling theory, \(\gamma/\nu = 2 - 2x_c = 5/3\).

In conclusion, we have obtained exact results at the critical point of the three-state antiferromagnetic Potts model. These results can be easily generalized to include next-nearest-neighbor interactions. The \(T = 0\) Potts model is then related to the general zero-field six-vertex model and results similar to Eqs. (2) and (3) follow. We note that measurement of finite-size-scaling amplitudes by Monte Carlo simulations can be useful in determining the nature of critical points, and also that knowledge of the finite-size-scaling amplitudes can be used to speed up convergence of numerical data.\(^7\)

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Hyunggyu Park and Mike Widom
Department of Physics
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213

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