# Topological Phase Transitions

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Abstract

We use methods of statistical mechanics to study the Julia set of the mapping $f(z) = z^2 + p$. For most values of $p$ these methods allow extremely accurate determination of the fractal dimension and escape rate. At special values of $p$ the Julia set undergoes a change in topology. We study the value $p = 1/4$ in detail and find singularities which we interpret as a phase transition.

## 1. Introduction

Julia sets are unstable invariant sets of complex analytic mappings. They possess chaotic dynamical behavior, a fractal dimension, and sometimes sensitive dependence on parameters. Despite these remarkable properties, which are shared by strange attractors, the theory of Julia sets is well established in an elegant and extensive mathematical literature.

In this article we consider the fractal dimension and escape rate of Julia sets. We utilize theorems of RUELLE [1] and BOWEN [2] in numerical measurements of these quantities. In this respect this article is a refinement of an earlier paper [3] which discussed the Julia set of

$$f(z) = z^2 + p \tag{1.1}$$

in the limits of large and small $p$. The statistical mechanical description of Julia sets [4,5] allows extremely accurate measurements of the fractal dimension and escape rate when $p$ is small.

The present article is also an extension of [3] because we explore a new range of values of the parameter $p$. In particular, we study phenomena on and near the boundary of the Mandelbrot set [6]. The Mandelbrot set, $M$, is the set of all values of $p$ for which Julia set of (1.1) is connected. This is the set of parameters for which the critical point does not iterate to infinity [7].

Consider a subset of $M$ consisting of the set of parameter values for which $f$ has a stable fixed point. The Julia set in this case is the boundary of the basin of attraction and is a Jordan curve. The critical point lies in the basin of attraction. In order to determine the parameter values which constitute this subset we compute the derivatives of the mapping at the fixed points

$$\lambda = 1 - \sqrt{1-4p} \tag{1.2}$$

$$\mu = 1 + \sqrt{1-4p} \quad . \tag{1.3}$$

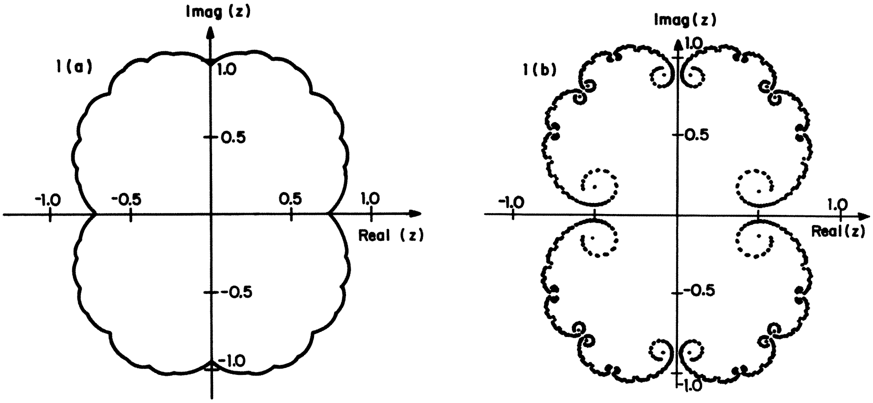


Figure 1. The Julia set of the mapping (1.1) with (a) $p = 0.20$, and (b) $p = 0.26$

The mapping (1.1) has a stable fixed point for any parameter value inside the cardioid

$$C = \{p : |\lambda(p)| < 1\} \quad . \tag{1.4}$$

Imagine varying the parameter $p$ in such a way that

$$\theta = \arg (\lambda) \tag{1.5}$$

is held fixed while $|\lambda|$ increases. In this article we concentrate on the value $\theta = 0$. As $p$ increases from 0 to 1/4, $\lambda$ increases from 0 to 1. The value $p = 1/4$ lies on the boundary of $C$. For $p > 1/4$ the critical point iterates to infinity, and the Julia set is completely disconnected. Thus the value $p = 1/4$ lies on the boundary of $M$ and as the parameter crosses this boundary the Julia set undergoes a transition of its topology (see Figure 1).

## 2. Small p

Many authors have noted an analogy between a statistical mechanical system on a one-dimensional lattice and a mapping [5,8-11]. In this analogy a bond on the lattice corresponds to an iteration of $f$, the state space at each lattice site corresponds to the domain and range of $f$, and the interaction between lattice sites can be represented by a transfer operator with kernel

$$T_D(z',z) = \delta^{(2)}(z' - f(z)) \left|\frac{df}{dz}\right|^{2-D} \tag{2.1}$$

where $\delta^{(2)}$ is the two-dimensional delta function, and $D$ is a parameter which will play the role of inverse temperature.

The partition function on a lattice with $N$ sites and periodic boundary conditions is

$$Z_N = \int dz T_D^N(z,z) = \sum_{\mathrm{Fix}\, f^N} \frac{\left|\frac{df^N}{dz}\right|^{2-D}}{\left|1 - \frac{df^N}{dz}\right|^2} \tag{2.2}$$

We choose to restrict the integration in (2.2) so that the sum includes only unstable cycles. RUELLE [4] has shown that the spectrum of $T$ is discrete and has a real nondegenerate largest eigenvalue, provided that $p$ is not on the boundary of $M$. Thus

$$Z_N(p,D) = \sum_{m=0}^{\infty} \lambda_m^N \tag{2.3}$$

where $\lambda_0 > 0$ and $\lambda_0 > |\lambda_1| \geqslant |\lambda_2| \geqslant \cdots$ . In addition RUELLE [1] showed that the $\lambda_m$ are real analytic functions of $D$ and $p$.

This theorem possesses enormous utility in numerical computations of the fractal dimension and escape rate for small $p$. The fractal dimension is defined by [1,2]

$$\lambda_0(D_F) = 1 \tag{2.4}$$

and the D-dimensional escape rate, $R$, is defined by [3]

$$R = \log \lambda_0(D) \quad . \tag{2.5}$$

Our numerical approach is to compute the partition function $Z_N$ for several values of $N$ and fit this to a truncated sum of exponentials as in (2.3). Remarkably we find that the eigenvalues $\lambda_m$ are all real for $0 \leqslant p < 1/4$. Thus we can fit $L$ eigenvalues exactly to $L$ values of $Z_N$. We have carried out this procedure for lattices of up to $N = 12$ sites. In Table 1 we show the value of $D$ for which $\lambda_0(D) = 1 \pm 10^{-13}$ as computed in an $L = 1,2,3,4$ eigenvalue fit.

Table 1. Convergence of $D_F$ when $p = 0.025$

| $L$ | $D_F$ |
|---|---|
| 1 | 1.00021 |
| 2 | 1.00023457 |
| 3 | 1.0002345919 |
| 4 | 1.00023459189 |

## 3. Phase Transition

Notice that the derivative at the unstable fixed point, $\mu$, equals 1 when $p = 1/4$. This means that while thermodynamic averages may still exist, the partition function $Z_N$ becomes infinite. By choosing a more complicated transfer operator than (2.1) we can define a new partition function

$$\hat{Z}_N = \sum_{\text{Fix } f^N} \left|\frac{df^N}{dz}\right|^{-D} \quad . \tag{3.1}$$

When $p$ is not on the boundary of $M$ the cycles are all unstable and the sum in (3.1) equals the sum in (2.2) except for corrections which are exponentially small in relation to the partition function. RUELLE [4] has shown that when $p$ is not on the boundary of $M$

$$\hat{Z}_N = \hat{\lambda}_0^N \pm \hat{\lambda}_1^N \pm \hat{\lambda}_2^N \pm \cdots \tag{3.2}$$

where $\hat{\lambda}_0 = \lambda_0$. We propose to employ equations (3.1) and (3.2) when $p = 1/4$.

We can compute $\hat{\lambda}_m(D,p)$ in some special cases. Consider first $D = 0$, $0 \leqslant p \leqslant 1/4$.

There are $2^N - 1$ cycles of length $N$, so that

$$\hat{Z}_N(D,p) = 2^N - 1 \quad . \tag{3.3}$$

Now consider the limit of large $D$. Only cycles with small $|df^N/dz|$ will contribute to the sum (3.1). The fixed point has the smallest derivative, thus

$$\hat{Z}_N(D,p) \approx \mu^{-DN} \tag{3.4}$$

so that $\hat{\lambda}_0 \approx (1 + \sqrt{1-4p})^D$ for large D. Aside from these special cases the computation must be done numerically.

We will now report the results of our numerical work. The convergence of some of our results is far worse than that shown in Table 1. We will discuss the reliability of our results and the possible cause of poor convergence. Two principal results of which we are quite confidant are

$$D_F(p) = D^* - x_0\sqrt{1-4p} + 0(1-4p) \tag{3.5}$$

where $D^* = 1.083$ and $x_0 = 0.15$, and

$$\hat{\lambda}_0 = (1 + \sqrt{1-4p})^{-D} + 0(1-4p) \tag{3.6}$$

when $D \geqslant D^*$. A third result, valid for $D < D^*$, is

$$\hat{\lambda}_0 = 1 + A(D^* - D)^y + 0(\sqrt{1-4p}) \tag{3.7}$$

where $A = -0.434$ and $y = 1$. Equations (3.6) and (3.7) show that $\hat{\lambda}_0$ is a singular function of $D$ at $D^*$. By analogy with statistical mechanics we will analyze this singularity as if it were a singluarity in some thermodynamic quantity as a function of inverse temperature. Thus, assume

$$\hat{\lambda}_0(D,p) - 1 = (D^* - D)^a \psi\left(\frac{D^*-D}{(1-4p)^b}\right) \quad . \tag{3.8}$$

Equation (3.6) requires $\psi(x) \sim x^{-a}$ for $x \to -\infty$ and $ab = 1/2$. Equation (3.7) requires $\psi(x) \sim A$ for $x \to +\infty$ and $a = y$. Assume that $x_0$ is a zero of $\psi(x)$. By equations (2.4) and (3.8) we have

$$D_F(p) = D^* - x_0(1-4p)^{1/2y} \tag{3.9}$$

which is consistent with (3.5) if $y = 1$.

## 4. Numerical Difficulties

Our numerical procedure is similar to that used to study small p. The essential difference is that we must allow subtraction as well as addition of eigenvalues in fitting $\hat{Z}_N$. Padé analysis on the zeta function [1,4] leads us to conclude that the spectrum is still discrete, real, and positive, and that the partition function takes the form

$$\hat{Z}_N = \hat{\lambda}_0^N + 1 - \hat{\lambda}_2^N + \hat{\lambda}_3^N - \hat{\lambda}_4^N + \cdots \quad . \tag{4.1}$$

Equation (4.1) suggests that the measurement of $D^*$ should improve exponentially with $L$ as we saw in Section 2. In Table 2 we see that instead we have $1/N$ type corrections. Our preliminary evidence based on fits to (4.1) shows that when $D < D_F$ the value of $\hat{\lambda}_0$ converges exponentially while all other $\hat{\lambda}_m$ approach 1 with $1/N$ corrections. When $D > D_F$ all eigenvalues approach 1 with $1/N$ corrections.

Table 2. Convergence of $D_F$ when $p = 1/4$. The fit is to (4.1) and N is the largest size lattice used in the fit. The third column includes corrections up to $(1/N)^3$

| N | $D_F$ | $D_F$ corrected |
|---|---|---|
| 14 | 1.073217 | 1.095506 |
| 15 | 1.073734 | 1.089383 |
| 16 | 1.074229 | 1.086428 |
| 17 | 1.074688 | 1.084867 |
| 18 | 1.075107 | 1.083814 |
| 19 | 1.075487 | 1.083152 |

These numerical difficulties lead us to speculate that the spectrum may actually be continuous between 0 and 1 with an isolated $\hat{\lambda}_0 > 1$ when $D < D^*$. The Padé analysis could easily miss this fact. It is interesting to note that $1/N$ decays have been observed in other dynamical systems which are hyperbolic except at isolated points [12].

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