

RENORMALIZATION GROUP ANALYSIS OF BIFURCATIONS IN AREA-PRESERVING MAPS

Michael WIDOM and Leo P. KADANOFF

The James Franck Institute and The University of Chicago Chicago, IL 60637, USA

Received 16 November 1981

Revised 26 February 1982

Two-dimensional area-preserving maps can be represented by a generating function, the action. High orders of successive period-doubling bifurcations are studied by writing a renormalization group scheme for this action. Fixed points and eigenvalues of this scheme are found and interpreted.

1. Introduction

We study the universal properties of period-doubling bifurcation sequences in two-dimensional area-preserving maps. This paper begins with a description of the period-doubling process based on the work of Greene, MacKay, Vivaldi, and Feigenbaum [1] and others [2, 3, 4]. The scaling laws which the period-doubling process is found to obey are explained through a renormalization argument previously presented by Greene et al. [1], by Helleman [5, 6] and others [7, 8]. This argument assumes the existence of a universal mapping which reproduces itself under a combined linear change of coordinates and composition with itself. We determine this function numerically, utilizing the principle of least action to perform the compositions. The effects of perturbations on this universal mapping are analyzed.

2. Period doubling in two dimensions

Greene et al. [1], consider a family of mappings

$$T_p: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -y + f_p(x) \\ x - f_p(x') \end{pmatrix}, \quad (1)$$

where

$$f_p(x) = px - (1-p)x^2. \quad (2)$$

These mappings can be factored into a product of two involutions, $T_p = I_2 I_1$, where

$$I_1: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}, \quad (3)$$

$$I_2: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y + f_p(x) \\ x - f_p(x') \end{pmatrix}. \quad (4)$$

The line $y = 0$ is invariant under I_1 and is called the dominant symmetry line. The existence of this line simplifies tremendously the process of locating cycles.

The work of refs. 1 through 4 shows that there is a sequence of cycles of length 2^N each with precisely two points on the dominant symmetry line. The bifurcation to a 2^{N+1} cycle occurs when one of the two points initially on the line becomes unstable creating a pair of stable points on opposite sides of the symmetry line. Simultaneously the other point on the symmetry line splits into two points both still lying on the symmetry line. In fig. 1 we show two unstable cycle elements on the symmetry line. Four elements of the stable cycle are shown, two near each unstable point.

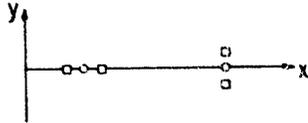


Fig. 1. Distribution of cycle elements in the x - y plane. Circles indicate an unstable 2-cycle whereas squares indicate a stable 4-cycle.

Let T_N denote the mapping produced by iterating T 2^N times. Each element of the 2^N cycle is a fixed point of T_N . Two of these elements lie on the dominant symmetry line: $(X_{0,N}; 0) \equiv z_{0,N}$ and $(X_{1,N}; 0) \equiv z_{1,N}$. The operation T_{N-1} maps these points into each other. Following earlier workers [1, 2], we distinguish between these two points by noting what happens to them upon the bifurcation which produces the 2^{N+1} cycle. At the bifurcation $z_{0,N}$ splits into a pair of points $z_{0,N+1}$ and $z_{1,N+1}$ which still lie upon the dominant symmetry line, while the "daughters" of $z_{1,N}$ fall off this line. In fig. 2 we have plotted the locations of the two cycle elements $z_{0,N}$ and $z_{1,N}$ as a function of the parameter p . Two crucial facts which can be explained through a renormalization argument are:

(a) the parameter values p_N where the 2^N

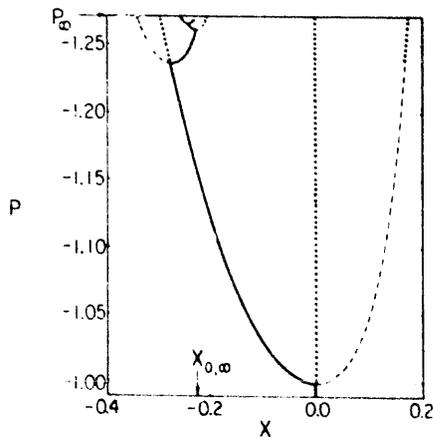


Fig. 2. Location of cycle elements on dominant symmetry line as a function of parameter. Dotted lines indicate unstable cycle elements, dashed lines indicate stable cycle elements which as p is further increased will bifurcate off the symmetry line (and off our picture), and solid lines indicate the principal point which will bifurcate along the symmetry line. See Greene et al. [1] for a further explanation of an essentially similar picture.

cycle becomes unstable converge geometrically to a finite value p_∞ in the limit of large N .

$$(p_N - p_\infty) \sim \delta^{-N}; \tag{5}$$

(b) when $p = p_\infty$ the separation of cycle elements along the line $y = 0$ vanishes geometrically. If we define

$$D_N = x_{1,N} - x_{0,N}, \tag{6}$$

then in the limit of long cycles

$$D_N \sim \alpha^{-N}; \tag{7}$$

(c) the behavior is generic in the sense that almost all choices of T lead to the same large N behavior. In particular the index values Greene et al., find for the map of eqs. (1)

$$\alpha = -4.018076704,$$

$$\delta = 8.721097200,$$

are expected to apply to almost all maps of the form (1).

The scaling equations (5) and (7) can be explained through a renormalization group scheme [1, 5-8] similar to that used by Feigenbaum [9] in the case of one-dimensional mappings. The fixed point equation for the renormalization group which we study here takes the form

$$T^* \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} T^* \circ T^* \begin{pmatrix} u_0 + x/\alpha \\ y/\beta \end{pmatrix}. \tag{8}$$

We solve this equation numerically. For the remainder of this paper we discuss our solution and the effect of perturbations.

3. Action principle

Area-preserving maps are of interest because they generally describe the time evolution of a

Hamiltonian system through a Poincaré surface of section. When working with such systems it is common to determine the motion through the principle of least action rather than directly through the equations of motion. When solving eq. (8) the action provides a means of insuring that the mapping is area preserving and thus removing one of the "relevant" perturbations. The action principle becomes especially simple when the Poincaré surface of section is employed since the path of least action is reduced to a discrete set of points.

Suppose the path starts at (\bar{x}) , passes through (\bar{x}') , and ends at (\bar{x}'') . Each step has an action associated with it, and the action for the entire trip is the sum of the actions for each step. If we choose the initial and final x values as our canonical variables then we have

$$A_1(x, x'') = A_0(x, x') + A_0(x', x''). \tag{9}$$

The principle of least action demands that A_1 is extremal with respect to the intermediate point x' . Thus we must have

$$\frac{\partial A_0(x, x')}{\partial x'} + \frac{\partial A_0(x', x'')}{\partial x'} = 0. \tag{10}$$

We can satisfy eq. (10) automatically if we assume that the action is the generating function satisfying

$$y = \frac{-\partial A_0(x, x')}{\partial x}, \quad y' = \frac{\partial A_0(x, x')}{\partial x'}. \tag{11}$$

This representation in terms of an action has several advantages over the original mapping. First, the area-preserving property is satisfied automatically,

$$\begin{aligned} \frac{\partial(x', y')}{\partial(x, y)} &= \frac{\partial(x', y')}{\partial(x, x')} \bigg/ \frac{\partial(x, y)}{\partial(x, x')} \\ &= \frac{-\partial^2 A}{\partial x \partial x'} \bigg/ \frac{-\partial^2 A}{\partial x' \partial x} = 1. \end{aligned} \tag{12}$$

Second, composing the mapping is equivalent to simply adding the actions. Thus if we wish to determine the action for the composed mapping $T^2 = T \circ T$ we just form $A_1(x, x'') = A_0(x, z) + A_0(z, x'')$ where $z = z(x, x')$ makes the right-hand side extremal.

The action formulation can also be used to find generating functions for T_N . Define A_0 as above and let $A_{N+1}(x, x'')$ be the extremum in x' of $A_N(x, x') + A_N(x', x'')$. Then, A_{N+1} is a generating function which describes 2^{N+1} applications of T .

Notice also that $A_N(x, x'')$ can be used to find elements of cycles of order 2^{N-q} for q being a non-negative integer. These cycle elements, x^* , are all fixed points of T_N . They can all be determined by considering

$$U_N(x) = A_N(x, x).$$

These fixed points all obey the condition

$$\left. \frac{\partial U_N(x)}{\partial x} \right|_{x^*} = y^* - y^* = 0$$

and hence are all extrema of U_N .

4. The universal action

We calculate the universal action starting from a modified form of the mapping in eq. (1). We take

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T_{p_n} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{13}$$

so that, if we define $f(\cdot) = f_{p_n}(\cdot)$

$$x' = y + f(x), \quad y' = -x + f(x'). \tag{14}$$

To determine the action we solve the equations

$$\begin{aligned} \frac{\partial A_0(x, x')}{\partial x} &= -y = f(x) - x', \\ \frac{\partial A_0(x, x')}{\partial x'} &= y' = f(x') - x \end{aligned} \tag{15}$$

and find

$$A_0(x, x') = -xx' + G(x) + G(x'), \quad (16)$$

where

$$\frac{\partial G(x)}{\partial x} = f(x). \quad (17)$$

Thus our initial action is symmetrical in x and x' , and all composed actions will also be symmetrical. This fact allows us to specify the action numerically with roughly half the effort required in the absence of this symmetry. As a further benefit an unstable eigenvalue is projected out (see section 5). Note that the modification leaves the line $y = 0$ invariant so that the dominant symmetry line is unchanged.

Several equivalent representations of the action are employed in our computation. We form a grid of points (x_μ, x'_ν) , where x_μ and x'_ν take on the values $(-1, -0.8, \dots, 0.6, 0.8, 1)$, and evaluate the action at each of these points. As our initial representation of the action we form the polynomial in x and x' which takes on the value of the action everywhere on the grid. In this representation our action is an array of coefficients A_{ij} with

$$A(x_\mu, x'_\nu) = \sum_{ij} A_{ij} x_\mu^i x'_\nu^j. \quad (18)$$

We compose the action with itself by applying eq. (9) with the initial and final points lying on the grid. Since the intermediate point must be varied continuously to extremize the action, we use expressions which are polynomials in the intermediate point. This is accomplished by defining the two functions $A1$ and $A2$. $A1$ is a set of polynomials in x' , each polynomial being essentially eq. (18) with x_μ on the grid but x' a continuous variable. Thus

$$A(x_\mu, x') = \sum_j A1_j(x_\mu) x'^j, \quad (19)$$

where

$$A1_j(x_\mu) = \sum_m A_{mj} x_\mu^m. \quad (20)$$

$A2$ is the analogous function with the first variable continuous and the second on the grid. We then add these actions together to form

$$\begin{aligned} A3_{\mu\nu}(z) &= A(x_\mu, z) + A(z, x'_\nu) \\ &= \sum_m [A1_m(x_\mu) + A2_m(x'_\nu)] z^m. \end{aligned} \quad (21)$$

For each (μ, ν) we find $z(x_\mu, x'_\nu)$ which extremizes the eq. (21). Now, according to eqs. (9) and (10), we can determine the value of the composed action everywhere on the grid: $[A \circ A](x_\mu, x'_\nu) = A3_{\mu\nu}(z(x_\mu, x'_\nu))$. As a final step we convert the composed action to the polynomial representation.

To determine the universal function we start with the action given by eqs. (16) and (17) which should differ from the universal function only by irrelevant perturbations since $p = p_*$. We shift our coordinate system so that the origin lies at the two-cycle element $x_{0,1}$. We then scale the coordinate system so that our grid covers $x_{0,2}$, but not $x_{1,1}$. This is necessary because we interpret $z(x, x')$ as the intermediate point through which the trajectory must pass in travelling from x to x' in two steps. It is not possible, however, to travel from $x_{0,1}$ to $x_{1,1}$ in two steps, so $z(x, x')$ cannot be defined for too large a grid. Now we follow eqs. (18) through (21) to compose this action. Note that this composed action, A_1 , has a polynomial representation whose linear terms vanish since the origin is a fixed point of the composed mapping.

At this point we begin to look for the universal action. If we shift coordinates so that the four-cycle element $x_{0,2}$ lies at the origin, rescale as discussed above, and perform the composition, we find that the new action A_2 is "close" to the old action A_1 . Aside from the unimportant constant term, A_2 is approximately

Table I
The negative of the coefficients A_{ij} in the polynomial for the action $A^*(x, x') = \sum_{ij} A_{ij} x^i x'^j$

$I \backslash J$	0	1	2	3	4	5
0	0.00000	0.00000	0.63588	-0.76529	0.00161	0.00025
1	0.00000	1.00000	0.06168	0.00876	0.00144	0.00027
2	0.63588	0.06168	0.01393	0.00315	0.00073	0.00018
3	-0.76529	0.00876	0.00315	0.00101	0.00031	0.00009
4	0.00161	0.00144	0.00073	0.00031	0.00012	0.00004
5	0.00025	0.00027	0.00018	0.00009	0.00004	0.00002

proportional to A_1 . Thus we have an approximate solution of

$$A^*(x, x') = \beta\alpha \left\{ A^* \left(u_0 + \frac{x}{\alpha}, u_0 + \frac{z}{\alpha} \right) + A^* \left(u_0 + \frac{z}{\alpha}, u_0 + \frac{x'}{\alpha} \right) \right\} \equiv \mathcal{R}[A^*(x, x')]. \tag{22}$$

To improve our solution we varied u_0, β, α , and the coefficients of the action polynomial to minimize the difference between the left- and right-hand sides of eq. (22). We achieved agreement of at least 10^{-10} in all quantities. In agreement with ref. 1, we find $\alpha = -4.018076704$ and $\beta = 16.36389688$. Low order terms in our action are given in table I.

Shifting by $u_0 = 0.286849$ is a nonuniversal feature of our renormalization equation (22). This can easily be eliminated through the use of a coordinate system with $z_{0,\infty}$ at the origin. Thus $A'(x, x') = A^*(s + x, s + x')$ with $s = u_0/(1 - 1/\alpha)$ obeys the renormalization equation

$$A'(x, x') = \beta\alpha \left\{ A' \left(\frac{x}{\alpha}, \frac{z}{\alpha} \right) + A' \left(\frac{z}{\alpha}, \frac{x'}{\alpha} \right) \right\}.$$

5. Perturbations of the action

It is important to know what happens if we have an action slightly different from A^* . Some perturbations are relevant and grow under iteration so that their presence will destroy the universal period-doubling behavior. Other per-

turbations are irrelevant and vanish under iteration so that their presence will not alter the universal behavior. Still others neither grow nor vanish and are called marginal. Universality follows from observing that there are a small number of relevant perturbations and no non-trivial marginal perturbations. Thus all mappings converge to the universal mapping provided a small number of parameters are adjusted to eliminate the relevant perturbations.

We assume that all perturbations can be described in terms of a complete set of eigenperturbations which have the property

$$(A^*(x, x') + \epsilon P_i(x, x')) - \mathcal{R}(A^*(x, x') + \epsilon P_i(x, x')) = -\lambda_i \epsilon P_i(x, x'). \tag{23}$$

For small perturbations eq. (23) reduces to

$$P_i(x, x') - \beta\alpha \left[P_i \left(u_0 + \frac{x}{\alpha}, u_0 + \frac{z}{\alpha} \right) + P_i \left(u_0 + \frac{z}{\alpha}, u_0 + \frac{x'}{\alpha} \right) \right] = -\lambda_i P_i(x, x'), \tag{24}$$

where z is the extremal point defined in eq. (22).

Several perturbations and eigenvalues can be identified easily.

1. The trivial perturbation $P(x, x') = \text{constant}$ grows with eigenvalue $\lambda = 2\beta\alpha$.

2. If we consider $A^*(x + \epsilon g(x), x' + \epsilon g(x'))$, with $g(x) = x^N$, we find that

$$P_N(x, x') = x^N \frac{\partial A^*(x, x')}{\partial x} + x'^N \frac{\partial A^*(x, x')}{\partial x'}$$

Table II
Leading eigenvalues λ_i
which define the growth
rate for perturbations of A^*

Eigenvalues	
-131.502785728	$2\beta\alpha$
8.721097206	δ
-4.018076706	α
1.000000000	α'
-0.248875288	α^{-1}
-0.116629426	
0.072984281	
0.061938931	
-0.015415103	
0.003836360	

is an eigenperturbation with eigenvalue $\lambda_N = \alpha^{1-N}$. Note that $\lambda_0 = \alpha$ is relevant and corresponds to a shift in x .

3. There are two marginal perturbations with $\lambda = 1$. One of these is $P_1(x, x')$ which corresponds to a dilation of the x -coordinate. There is a corresponding dilation of the y -axis which is also marginal. These marginal perturbations are a consequence of the fact that we can replace $A^*(x, x')$ with $bA^*(ax, ax)$ and still solve eq. (22).

4. The perturbation $Q_1(x, x') = x - x'$ breaks the symmetry of the action $A^*(x, x') = A^*(x', x)$, and corresponds to a shift in y with eigenvalue $\lambda = \beta$.

5. If we were to start with $p \neq p_\infty$ we could only see a fraction of the bifurcation series. Therefore there must be a relevant perturbation with eigenvalue δ corresponding to a change in parameter. The exact form of this perturbation cannot be determined a priori but emerges as a result of the perturbation analysis.

6. The eigenvalue $\lambda = -0.117$ is observed in the convergence of the residue [1] to its universal value [11].

Table II lists several of the eigenvalues obtained from linearizing R around the fixed point A^* and only considering perturbations for which $P(x, x') = P(x', x)$.

Acknowledgements

We would like to thank J. Greene, R.S. MacKay, M. Feigenbaum, F. Vivaldi, and R. Helleman for helpful discussions and/or sending us copies of their work prior to publication. We further thank Greene and MacKay for commenting upon a draft of this paper. In the course of this work we have had frequent and helpful discussions with Albert B. Zisook and Scott J. Shenker. This work was supported in part by the Materials Research Laboratory Program of the National Science Foundation at the University of Chicago under Grant 79-24007.

References

- [1] J.M. Greene, R.S. MacKay, F. Vivaldi and M.J. Feigenbaum, *Physica 3D* (1981) 468.
- [2] T.C. Bountis, *Physica 3D* (1981) 557.
- [3] J. Van Zeyts, figs. 13 and 14 in ref. 5.
- [4] G. Benettin, C. Cercignani, L. Galgani and A. Giorgillo, *Lett. Nuovo Cimento* 28 (1980) 1, 29 (1980) 163.
- [5] R.H.G. Helleman, in *Fundamental Problems in Statistical Mechanics*, vol. 5, E.D.G. Cohen, ed. (North-Holland, Amsterdam, 1980), p. 165.
- [6] R.H.G. Helleman and R.S. MacKay, in *Nonequilibrium Problems in Statistical Mechanics*, W. Horton, L. Riechl and V. Szebehely, eds. vol. 2 (Wiley, New York, 1981).
- [7] B. Derrida and Y. Pomeau, *Phys. Lett.* 80A (1980) 217.
- [8] P. Collet, J.P. Eckmann and H. Koch, *Physica 3D* (1981) 457.
- [9] M.J. Feigenbaum, *J. Stat. Phys.* 19 (1978) 25; 21 (1979) 669.
- [10] A.B. Zisook, preprint.
- [11] R.S. MacKay, private communication.