Parallel flow in Hele-Shaw cells with ferrofluids

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(September 14, 1999)

Abstract

Parallel flow in a Hele-Shaw cell occurs when two immiscible liquids flow with relative velocity parallel to the interface between them. The interface is unstable due to a Kelvin-Helmholtz type of instability in which fluid flow couples with inertial effects to cause an initial small perturbation to grow. Large amplitude disturbances form stable solitons. We consider the effects of applied magnetic fields when one of the two fluids is a ferrofluid. The dispersion relation governing mode growth is modified so that the magnetic field can destabilize the interface even in the absence of inertial effects. However, the magnetic field does not affect the speed of wave propogation for a given wavenumber. We note that the magnetic field creates an effective interaction between the solitons.

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Systems of immiscible fluids often exhibit hydrodynamic instabilities leading to pattern formation. The Saffman-Taylor problem [1] considers two immiscible viscous fluids moving in a narrow space between two parallel plates (the so-called Hele-Shaw cell). When a low viscosity fluid invades a region filled with high viscosity fluid, the initially flat fluid-fluid interface is unstable and evolves through a mechanism known as viscous fingering [2]. We call the displacement of one fluid by another *frontal flow*. In contrast, *parallel flow* occurs when the fluids flow parallel to the interface separating them. One important example of parallel flow occurs after the passage of a fully developed Saffman-Taylor finger. Far downstream from the finger tip the fluids flow parallel to their interface.

Recent experimental and theoretical studies [3–5] examined the dynamics of fluid interfaces under parallel flow in Hele-Shaw cells. Zeybek and Yortsos [3,4] studied, both theoretically and experimentally, parallel flow in a horizontal Hele-Shaw cell in the large capillary number limit. For finite capillary number and wavelength, linear stability analysis indicates that small perturbations decay, but the rate of decay vanished in the limit of large capillary numbers and large wavelength. Furthermore, a weakly nonlinear analysis of the problem found Korteweg-de Vries (KdV) dynamics leading to stable finite amplitude soliton solutions. Solitons were indeed observed experimentally. Gondret and Rabaud [5] incorporated inertial terms into the equation of motion in a Hele-Shaw cell and found a Kelvin-Helmholtz instability for inviscid fluids. For viscous fluids they derived a Kelvin-Helmholtz-Darcy equation and found the threshold for instability was governed by inertial effects, while the wave velocity was governed by the Darcy's law flow of viscous fluids. Their experimental results supported their theoretical analysis.

As was the case for frontal flow of nonmagnetic fluids in Hele-Shaw cells, many research groups have studied the frontal interface behavior when one of the fluids is a ferrofluid [6], and an external magnetic field is applied [7–10]. Ferrofluids, which are colloidal suspensions of microscopic permanent magnets, respond paramagnetically to applied fields. As a result of the ferrofluid interaction with the external field, the usual frontal displacement viscous fingering instability is supplemented by a magnetic fluid instability, resulting in a variety of new interfacial behaviors. Depending on the applied field direction, one observes highly branched, labyrinthine structures [7,8], patterns showing an ordered line of peaks [9], or even the supression of the usual viscous fingering instability [10].

In this paper we perform the linear stability analysis for parallel flow in which one fluid is a ferrofluid and a magnetic field is applied. We consider three separate field configurations: (a) *tangential*, for in-plane fields tangent to the unperturbed interface; (b) *normal*, for in-plane applied fields normal to the unperturbed interface; (c) *perpendicular*, when the field is perpendicular to the plane defined by the Hele-Shaw cell plates. We show the magnetic field provides additional mechanisms for destabilizing the interface, and we analyze qualitatively the interactions between solitons caused by the magnetic field. We neglect inertial terms because they are not needed to understand the interfacial instability.

Let us briefly describe the physical system of interest. Consider two semi-infinite immiscible viscous fluids, flowing with velocities U_1 and U_2, along the x direction, in a Hele-Shaw cell of thickness b (see figure 1). We assume that b is smaller than any other length scale in the problem, and therefore the system is considered to be effectively two-dimensional. Denote the densities and viscosities of the lower and upper fluids, respectively as ρ_1, η_1 and ρ_2, η_2. To achieve steady-state parallel flow the velocities and viscosities must obey the condition $\eta_1 U_1 = \eta_2 U_2$.

Between the two fluids there exists a surface tension σ. We assume that the lower fluid is the ferrofluid (magnetization $\vec{M}$), while the upper fluid is nonmagnetic. In order to include the acceleration of gravity $\vec{g}$, we tilt the cell so that the y axis lies at angle β from the vertical direction. To include magnetic forces, we apply a uniform magnetic field $\vec{H}_0$, which may point along the x, y or z axis. During the flow, the fluid-fluid interface has a perturbed shape described as $y = \zeta(x,t)$ (solid curve in figure 1).

Hydrodynamics of ferrofluids departs from the usual Navier-Stokes equations through the inclusion of a term representing magnetic force. Let $\vec{M}$ represent the local magnetization of the ferrofluid, and note that the force on $\vec{M}$ depends on the gradient of local magnetic field $\vec{H}$. The local field differs from the applied field $\vec{H}_0$ by the demagnetizing field of the polarized ferrofluid. We will assume $\vec{M}$ takes a constant value parallel to the applied field. This amounts to neglecting the demagnetizing field relative to the applied field and can be justified for very thin ferrofluid films, for low magnetic susceptibility of the ferrofluid, or for large applied fields that saturate the ferrofluid magnetization.

For the quasi two-dimensional geometry of a Hele-Shaw cell, the three dimensional flow may be replaced with an equivalent two-dimensional flow $\vec{v}(x,y)$ by averaging over the z direction perpendicular to the plane of the Hele-Shaw cell. Imposing no-slip boundary conditions and a parabolic velocity profile one derives Darcy's law for ferrofluids in a Hele-Shaw cell [11,12],

$$\eta \vec{v} = -\frac{b^2}{12}\left\{\vec{\nabla} p - \frac{1}{b}\int_{-b/2}^{+b/2} (\vec{M}\cdot\vec{\nabla})\vec{H}\,dz - \rho(\vec{g}\cdot\hat{y})\hat{y}\right\}, \tag{1}$$

where p is the hydrodynamic pressure. Equation (1) describes nonmagnetic fluids by simply dropping the terms involving magnetization.

When the velocity field $\vec{v}$ is irrotational, it is convenient to rewrite equation (1) in terms of velocity potentials. We write $\vec{v} = -\vec{\nabla}\phi$, where ϕ defines the velocity potential. Likewise we introduce the scalar magnetic potential

$$\varphi = \int_{\mathcal{S}} \frac{\vec{M}\cdot\vec{n}'}{|\vec{r}-\vec{r}'|} d^2r' \tag{2}$$

where $\vec{H} = -\vec{\nabla}\varphi$. Here the unprimed coordinates $\vec{r}$ denote arbitrary points in space. The primed coordinates $\vec{r}'$ are integration variables within the magnetic domain $\mathcal{S}$, and d^2r'

denotes the infinitesimal area element. The vector $\vec{n}'$ represents the unit normal to the magnetic domain in consideration.

To study the interface dynamics, we evaluate equation (1) for each of the fluids on the interface, subtract the resulting equations from each other, and divide by the sum of the two fluids' viscosities to get the equation of motion

$$A\left(\frac{\phi_2+\phi_1}{2}\right)+\left(\frac{\phi_2-\phi_1}{2}\right)=\frac{b^2}{12(\eta_1+\eta_2)}\times \left\{\sigma\kappa+\frac{1}{b}\int_{-b/2}^{+b/2}(\vec{M}\cdot\vec{\nabla}\varphi)dz+(\rho_2-\rho_1)g\cos\beta\ y\right\}. \tag{3}$$

To obtain (3) we have used the pressure boundary condition $p_2-p_1=\sigma\kappa$ at the interface, where $\kappa=(\partial^2\zeta/\partial x^2)[1+(\partial\zeta/\partial x)^2]^{-3/2}$ denotes the interfacial curvature in the plane of the Hele-Shaw cell. The dimensionless parameter $A=(\eta_2-\eta_1)/(\eta_2+\eta_1)$ is the viscosity contrast.

We perturb the interface with a single Fourier mode

$$\zeta(x,t)=\zeta_k\exp(i(\omega t-kx)). \tag{4}$$

The velocity potential for fluid i, ϕ_i, must contain the uniform unperturbed flow U_i and a perturbed part that reflects the space and time dependence of ζ, obeys Laplace's equation $\nabla^2\phi_i=0$ and vanishes as $y\to\pm\infty$. The velocity potentials obeying these requirements are

$$\phi_1=\phi_{1k}\exp(|k|y)\exp(i(\omega t-kx))-U_1\ x, \tag{5}$$

and

$$\phi_2=\phi_{2k}\exp(-|k|y)\exp(i(\omega t-kx))-U_2\ x. \tag{6}$$

To conclude our derivation and close equation (3) we need additional relations expressing the velocity potentials in terms of the perturbation amplitudes. To find these, consider the kinematic boundary condition, which states that the normal components of each fluid's velocity at the interface equals the normal velocity of the interface itself [6], i.e.

$$\frac{\partial \zeta}{\partial t} = \left(\frac{\partial \zeta}{\partial x} \frac{\partial \phi_i}{\partial x} \right)_{y=\zeta} - \left(\frac{\partial \phi_i}{\partial y} \right)_{y=\zeta} . \tag{7}$$

Insert expression (4) for $\zeta(x,t)$ and (5) for ϕ_1 into the boundary condition equation (7) and solve for $\phi_{ik}(t)$ consistently to first order in ζ to find

$$\phi_{1k} = -\frac{i\omega\zeta_k}{|k|} + i\frac{k}{|k|}\, U_1 \zeta_k, \tag{8}$$

and a similar expression for ϕ_{2k}.

Substitute expression (8) for ϕ_{1k} and the related expression for ϕ_{2k} into equation of motion (3), and again keep only linear terms in the perturbation amplitude. This procedure eliminates the velocity potentials from equation (3), and we obtain the dispersion relation for growth of the perturbation $\zeta(x,t)$

$$\omega = k \left(\frac{\eta_1 U_1 + \eta_2 U_2}{\eta_1 + \eta_2} \right) - \frac{i|k|\sigma}{12(\eta_1 + \eta_2)} \left[N_B I_j(k) - (kb)^2 - (k_0 b)^2 \right] \tag{9}$$

where

$$N_B = \frac{2M^2 b}{\sigma} \tag{10}$$

is the magnetic Bond number and

$$k_0 = \sqrt{\frac{(\rho_1 - \rho_2) g \cos\beta}{\sigma}}. \tag{11}$$

The real part of ω is k times the phase velocity, and is the viscosity-weighted average of the two fluid velocities. Note that the magnetic field does not alter the phase velocity of the waves. The imaginary part of ω, which governs the exponential growth or decay of the wave amplitude, does include effects of the magnetic field. Exponential (unstable) growth occurs when the imaginary part of ω is negative. We point out that when there is no applied magnetic field ($N_B = 0$) our equation (9) agrees with the dispersion relation derived by Gondret and Rabaud [5] for the case in which the cell is vertical ($\beta = 0$) and by Zeybek and Yortsos [3,4] for the case in which the cell is horizontal ($\beta = \pi/2$).

Terms containing $I_j(k)$ originate from the Fourier transforms of

$$M^2 I_j(x) \equiv \frac{1}{b}\int_{-b/2}^{+b/2} M_j \frac{\partial \varphi}{\partial r_j}\, dz, \tag{12}$$

the magnetic contribution to equation (3). The subscript $j = x, y, z$ indicates the tangential, normal and perpendicular magnetic field configurations, respectively. For $\vec{M}$ in the x or y direction we can expand equation (12) to first order in ζ to obtain

$$I_x(x) = \int_{-\infty}^{\infty} dx'(x - x')\left[-\frac{\partial \zeta(x')}{\partial x'}\right] \bar{F}(x - x') \tag{13}$$

and

$$I_y(x) = \int_{-\infty}^{\infty} dx'[\zeta(x) - \zeta(x')]\bar{F}(x - x') \tag{14}$$

where

$$\bar{F}(x) \equiv \frac{1}{b}\int_{-b/2}^{+b/2}\int_{-b/2}^{+b/2} \frac{dz dz'}{[x^2 + (z - z')^2]^{3/2}} = \frac{2}{bx^2}[\sqrt{b^2 + x^2} - |x|]. \tag{15}$$

In contrast, for $I_z(x)$ the z integration inverts the derivative of φ with respect to z in equation (12) so that after integrating over y' and expanding to first order in powers of ζ, this term simplifies to

$$I_z(x) = \int_{-\infty}^{\infty} dx' \frac{2}{b}\left[\frac{1}{\sqrt{(x - x')^2}} - \frac{1}{\sqrt{(x - x')^2 + b^2}}\right][\zeta(x') - \zeta(x)]. \tag{16}$$

We obtain the specific forms for the $I_j(k)$'s corresponding to each particular field configuration by taking the Fourier transform of equations (13), (14), and (16). After some simple algebra we find the following expressions for the magnetic terms $I_j(k)$

$$I_x(k) = -2 \int_0^{\infty} \left(\frac{\sin\tau}{\tau}\right)[\sqrt{(kb)^2 + \tau^2} - \tau]\; d\tau, \tag{17}$$

$$I_y(k) = 4 \int_0^\infty \left(\frac{\sin\tau}{\tau}\right)^2 [\sqrt{(kb/2)^2 + \tau^2} - \tau]\ d\tau, \tag{18}$$

and

$$I_z(k) = 4 \int_0^\infty \sin^2\tau \left[\frac{1}{\tau} - \frac{1}{\sqrt{(kb/2)^2 + \tau^2}}\right]\ d\tau. \tag{19}$$

In the limits of small and large wavevector these Fourier transforms reduce to

$$I_x(k) \approx \begin{cases} -[(3/2 - C + \ln 2) - \ln kb](kb)^2 & kb << 1 \\ -\pi kb & kb >> 1 \end{cases} \tag{20}$$

$$I_y(k) \approx \begin{cases} [(2 - C + \ln 2) - \ln kb](kb)^2/2 & kb << 1 \\ \pi kb & kb >> 1 \end{cases} \tag{21}$$

$$I_z(k) \approx \begin{cases} [(1 - C + \ln 2) - \ln kb](kb)^2/2 & kb << 1 \\ \ln(kb/2) & kb >> 1, \end{cases} \tag{22}$$

where $C \approx 0.57721$ denotes Euler's constant [13]. Our results (17), (18) and (19) agree with similar kind of calculations related to *frontal* displacements in Hele-Shaw cell with ferrofluids [7–10].

The dispersion relation (9) is given for the case of systems with infinite extent along the y-axis. For finite extent L the algebraic dependence on wavevector k is modified by a first order rational function of $\sinh kL$ as shown by Zeybeck and Yortsos [3,4]. When kL is large this finite size correction dies off exponentially quickly. The magnetic integrals $I_j(k)$ likewise possess exponentially small finite size corrections.

Consider the stability of the fluid-fluid interface for the different field configurations. The initially flat interface is unstable to perturbations with wavenumber k when $N_B I_j(k) - (kb)^2 - (k_0 b)^2$ is positive. If the heavier fluid is below the lighter fluid, $(\rho_1 > \rho_2)$, then both gravity and surface tension stabilize the system and k_0 is real. Therefore, in the absence of applied magnetic field ($N_B = 0$), the temporal growth rate of any perturbation is negative and waves are damped. On the other hand, if the external magnetic field is nonzero, the stability of the interface will depend on the field's direction. Figure 2 illustrates how the magnetic terms (17), (18) and (19) vary with reduced wave number kb. Inspecting figure 2 and the imaginary part of the dispersion relation (9) we note that a tangent field configuration ($I_x(k) < 0$), makes the growth rate even more negative than when the field is absent. So a tangent external field has a stabilizing nature, reinforcing the effects of gravity and surface tension. In contrast, since $I_y(k)$ and $I_z(k)$ are both positive quantities, if a sufficiently strong magnetic field is applied normal to the fluid-fluid interface, or perpendicular to the cell plates, the growth rate may become positive, leading to a possible destabilization of the interface. We conclude that the magnetic field can destabilize the interface even in the absence of inertial effects.

A physical explanation for these behaviors can be given based on simple energetic grounds. An external applied magnetic field aligns the internal magnetic moments of the ferrofluid in the field direction. As a consequence, the dipole-dipole interaction is attractive when the field is parallel to the dipoles' separation (tangent field configuration), and repulsive when the field is orthogonal to such separation (normal or perpendicular field configurations). Therefore, for normal and perpendicular fields, the ferrofluid spreads out in the Hele-Shaw plane due to magnetic repulsion, leading to interface deformation. In contrast, for a tangent field, any interface perturbation tends to be stabilized, due to magnetic attraction.

In addition to the interface stability issue discussed above, it is interesting to ask how the magnetic filed acts on the motion of interfacial waves once they appear. In the following, we discuss the action of the applied magnetic field on the solitons that appear in parallel flow in Hele-Shaw cells. To treat the problem rigorously would require reproducing the analysis of Zeybek and Yortsos [3,4] that derived Airy and KdV equations from a weakly nonlinear analysis of the interfacial perturbations. Here we simply point out that the solitons may be considered as localized perturbations on the flat interface. When magnetic fields are present the solitons acquire net dipole moments equal to the magnetization of the fluid multiplied by the integrated area of the soliton.

Take the generic form of a KdV soliton,

$$u(x,t) = -\frac{c}{2}\text{sech}\left(\frac{\sqrt{c}}{2}(x-ct)\right), \tag{23}$$

written here in terms of the scaled time, position and height variables discussed in [4], where c is the speed of propogation. We define the scaled dipole moment of the soliton of speed c as

$$m(c) = \int_{-\infty}^{\infty} \vec{M} u(x,t) dx = -\sqrt{c}\pi\vec{M}. \tag{24}$$

In doing so, we neglect the magnetic field dependence of the shape of the soliton. We may consider the magnetic moment (24) as the leading, linear term in a perturbative series in powers of applied field, and expect a cubic correction due to the field-dependent soliton shape. As noted in [4], the actual profile in unscaled coordinates may be either positive or negative, and the dipole moment given here must be divided by the position and height rescaling factors to yield the true moment. True dipole moments m point parallel to the magnetization $\vec{M}$ when the soliton consists of excess magnetic fluid, and points opposite to $\vec{M}$ when the soliton consists of missing magnetic fluid.

Dipole interactions are long-ranged, falling off as $1/x^3$ for moments separated by a distance x. This contrasts with the fluid-dynamic interaction of solitons which decays exponentially with separation. An interesting additional feature of the dipole-dipole interaction is its variation with the relative orientation of dipole moments and the vector joining them. In the case of solitons with parallel moments $\vec{m}_1$ and $\vec{m}_2$ displaced from each other along the x axis, the interactions will be attracting, with strength $2m_1m_2$, when the magnetizations lie along the x axis (tangetial) but will be repelling, with strength m_1m_2 when the magnetizations lie along the y (normal) or z (perpendicular) axes.

In conclusion, we have performed the linear stability analysis for parallel flow in a Hele-Shaw cell when one of the fluids is a ferrofluid. We show that the magnetic field may provide a new mechanism for destabilizing the interface in the absence of inertial effects, and we determine the magnetic correction to the dispersion relations for three distinct field orientations. Finally, we suggest parallel flow of ferrofluids as a novel system in which to investigate soliton interactions.

ACKNOWLEDGMENTS

We thank J.C. Bacri for interesting discussions on the possibility of conducting parallel flow experiments with ferrofluids. J.A.M. would like to thank CNPq (Brazilian Agency) for financial support. This work was supported in part by the National Science Foundation grant DMR-9732567.

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FIGURE CAPTIONS

FIG. 1: Schematic configuration of the parallel flow geometry. The lower fluid is a ferrofluid of magnetization $\vec{M}$. The magnetization is assumed to be constant and parallel to the uniform magnetic field $\vec{H}_0$, which may be oriented along the x, y or z axis. All quantities shown in the figure are defined in the text.

FIG. 2: Variation of $I_j(k)$ as a function of kb for (a) tangential (equation (17)), (b) normal (equation (18)), and (c) perpendicular (equation (19)) magnetic field configurations.