Fluctuations of particles and holes

1. The occupancy fluctuation for a quantum state is defined as

$$\sigma_n^2 = \langle n^2 \rangle - \langle n \rangle^2.$$  

For fermions this can be simply expressed in terms of the average occupation probability $\langle n \rangle$ and the average vacancy probability $\langle h \rangle = 1 - \langle n \rangle$ (here $h$ stands for occupation by a “hole”). What is the expression? *Hint*: What can you say about the value of $n^2$ for a fermion?

2. Sketch $\sigma_n^2$ as a function of $\langle n \rangle$ over the full range of $\langle n \rangle$. Justify the shape and special values of this function on physical grounds.
3. In a gas of fermions with chemical potential $\mu$, consider a state of energy $E$. The probability to occupy this state is

$$\langle n \rangle = \frac{1}{e^{\beta(E-\mu)} + 1}.$$

Rewrite the vacancy probability $1 - \langle n \rangle$ as the probability $\langle h \rangle$ to occupy a hole state, and relate the energy $E_h$ and chemical potential $\mu_h$ of the hole to the corresponding properties of the particle. Is the hole a fermion, or a boson?

4. Compute Deserno’s “fillability” $\chi \equiv \partial \langle n \rangle / \partial \mu$. Do you see a connection between this result and your expression for $\sigma_n^2$ found in part 1? The relationship is a simple example of a general connection between fluctuations and susceptibilities.
5. Does a similar relation hold for bosons? *Hint:* Recall that $\sigma_n^2 = \langle n \rangle (1 + \langle n \rangle)$ for bosons.