Particles and holes

1. The occupancy fluctuation for a quantum state is defined as

\[ \sigma_n^2 = \langle n^2 \rangle - \langle n \rangle^2. \]

For fermions this can be simply expressed in terms of the average occupation probability \( \langle n \rangle \) and the average vacancy probability \( \langle h \rangle = 1 - \langle n \rangle \) (here \( h \) stands for occupation by a “hole”). What is the expression? **Hint:** What can you say about the value of \( n^2 \) for a fermion?

**Answer:** Since \( n = 0, 1 \) for fermions, we have \( n^2 = n \) and \( \langle n^2 \rangle = \langle n \rangle \). Hence

\[ \sigma_n^2 = \langle n \rangle (1 - \langle n \rangle) = \langle n \rangle \langle h \rangle. \]

2. Sketch \( \sigma_n^2 \) as a function of \( \langle n \rangle \) over the full range of \( \langle n \rangle \). Justify the shape and special values of this function on physical grounds.

**Answer:** The function vanishes at \( \langle n \rangle = 0, 1 \) because the values of \( n \) are pinned to their limits. It is symmetric between particles and holes, hence it is maximal at \( \langle n \rangle = 1/2 \). The value at \( \langle n \rangle = 1/2 \) is \( (1/2)^2 = 1/4 \) because the allowed values \( n = 0, 1 \) differ from \( \langle n \rangle \) by 1/2.

![Graph of \( \sigma_n^2 \) as a function of \( \langle n \rangle \)]
3. In a gas of fermions with chemical potential \( \mu \), consider a state of energy \( E \). The probability to occupy this state is

\[
\langle n \rangle = \frac{1}{e^{\beta(E-\mu)} + 1}.
\]

Rewrite the vacancy probability \( 1 - \langle n \rangle \) as the probability \( \langle h \rangle \) to occupy a hole state, and relate the energy \( E_h \) and chemical potential \( \mu_h \) of the hole to the corresponding properties of the particle. Is the hole a fermion, or a boson?

**Answer:** Recall the identity

\[
\frac{1}{x + 1} + \frac{1}{1/x + 1} = 1
\]

which implies that

\[
\langle h \rangle = 1 - \langle n \rangle = \frac{1}{e^{-\beta(E-\mu)} + 1} = \frac{1}{e^{\beta(E_h-\mu_h)} + 1}
\]

with \( E_h = -E \) and \( \mu_h = -\mu \). The hole is a fermion because it obeys Fermi statistics.

4. Compute Deserno’s “fillability” \( \chi \equiv \partial\langle n \rangle / \partial \mu \). Do you see a connection between this result and your expression for \( \sigma_n^2 \) found in part 1? The relationship is a simple example of a general connection between fluctuations and susceptibilities.

**Answer:** First note that

\[
1 - \langle n \rangle = \frac{e^{\beta(E-\mu)}}{e^{\beta(E-\mu)} + 1}.
\]

Now, taking the requested derivative, we find

\[
\chi = \beta \frac{e^{\beta(E-\mu)}}{(e^{\beta(E-\mu)} + 1)^2} = \beta \langle n \rangle \langle h \rangle = \beta \sigma_n^2
\]

5. Does a similar relation hold for bosons? **Hint:** Recall that \( \sigma_n^2 = \langle n \rangle (1 + \langle n \rangle) \) for bosons.

**Answer:** Yes, something rather similar happens. We know that

\[
\langle n \rangle = 1/(\exp(\beta (E - \mu)) - 1)
\]

for bosons. Taking the derivative

\[
\frac{\partial \langle n \rangle}{\partial \mu} = \beta \frac{e^{\beta(E-\mu)}}{(e^{\beta(E-\mu)} - 1)^2},
\]

and noting the identity

\[
\frac{1}{x - 1} + \frac{1}{1/x - 1} = -1.
\]

results in

\[
\chi = \beta \langle n \rangle (1 + \langle n \rangle) = \beta \sigma_n^2
\]
as expected.