The following questions address the Ising model with Hamiltonian

\[ H = -J \sum_{(ij)} \sigma_i \sigma_j - h \sum_i \sigma_i. \]

2. The following figure illustrates the exact zero-field magnetization \( M(T, h = 0) \) for the Ising model on a square lattice in 2D and a simple cubic lattice in 3D, and also the mean field theory solution for each lattice. Units are chosen so that \( J = k_B = 1 \). Label each of the curves and briefly justify your claims, addressing both the critical temperatures \( T_c \) and the critical exponents \( \beta \).
3. Consider the Ising model on an equilateral triangle with $N = 3$ spins at the vertices. Take the antiferromagnetic case with $J < 0$. Describe the ground state and state the entropy at $T = 0$.

4. Recall that the low temperature series for the square lattice Ising model with $N$ sites and magnetic field $h = 0$ is

$$Z_0 = 2e^{2N\beta J}\{1 + Ne^{-8\beta J} + \cdots\}.$$  

In the following you will keep only terms of similar order to the above. Each part that follows depends on your previous result, so be very careful! If you cannot complete an early step, you can introduce a symbol to represent your answer then proceed with the later steps.

(a) Explain briefly the meaning or origin of the leading factor of 2, the exponential prefactor $\exp(2N\beta J)$, and the term $N\exp(-8\beta J)$ in the series expansion.
(b) Recalculate the expansion of $Z$ for the case of positive magnetic field, $h > 0$.

(c) Obtain a series expansion for the free energy. *Please work in the limit of large size $N$, such that $N\beta h \gg 1$.***
(d) Calculate the magnetization $M$.

(e) Determine the zero-field susceptibility defined as

$$\chi_0 \equiv \lim_{h \to 0^+} \frac{\partial M}{\partial h}.$$ 

Sketch your result for $\chi_0(T)$ and discuss its approach to its low temperature limit.