1. Recall that heat capacity is the rate at which a material gains energy as the temperature rises. Answer the following questions in a few brief sentences using only physical reasoning without any equations.

(a) Which has the greater heat capacity at low temperature, a Fermi gas or a Bose gas? Both have the same number of particles and the same masses in the same volumes.

**Answer:** Excitations of the Fermi gas occur at the Fermi energy, while excitations of the Bose gas occur at energies proportional to the temperature. Because the density of states vanishes at low energy but remains finite at the Fermi energy, the Fermi gas has the greater heat capacity.

(b) The heat capacity of an ideal gas at high temperature is $3k_B/2$, while the heat capacity of a harmonic solid is $3k_B$ at high temperature. Why is the heat capacity greater for a solid than for an ideal gas?

**Answer:** A harmonic solid stores energy both as potential energy in its chemical bonds (e.g. springs in a mass and spring model) and also as kinetic energy in its atomic motion. The ideal gas can only store energy as kinetic energy of atomic motion.
2. A donor state is the state of an electron bound to a donor ion in a semiconductor (e.g. an electron bound to a phosphorous ion in a doped semiconductor). The energy of this state, $E_d$, lies in the gap, slightly below the conduction band minimum, $E_d \lesssim E_c$. A donor state can be empty, or it can be occupied by a spin up electron or by a spin down electron. Because of Coulomb repulsion it cannot be occupied by a spin up and spin down electron at the same time.

(a) Write down the partition function $Z_d$ for a single donor state. Don’t forget to include the chemical potential $\mu$ of the electron.

**Answer:** Summing over the possible states (empty, spin up, spin down) yields

$$Z_d = \sum_{\text{empty, up, down}} e^{-\beta(E-\mu)n_d} = 1 + 2e^{-\beta(E_d-\mu)}.$$

(b) Determine the mean occupation $\langle n_d \rangle$ of the donor state.

**Answer**

$$\langle n_d \rangle = \sum_{\text{empty, up, down}} n_d e^{-\beta(E-\mu)n_d}/Z_d$$

$$= 2e^{-\beta(E_d-\mu)}/(1 + 2e^{-\beta(E_d-\mu)})$$

$$= \frac{1}{\frac{1}{2}e^{\beta(E_d-\mu)} + 1}.$$

(c) As a simplified model for a semiconductor, assume there are $N_d$ donor states all at energy $E_d$, and $N_c$ conduction states, all at a common energy $E_c$, with $E_d < E_c$. Express the total number of electrons $N_e$ in terms of the quantities $N_d, N_c, E_d, E_c$, and $\mu$.

**Answer:** The total number of electrons is the sum of the number in the donor states and the conduction states, hence

$$N_e = N_d = \frac{N_d}{\frac{1}{2}e^{\beta(E_d-\mu)} + 1} + \frac{N_c}{e^{\beta(E_c-\mu)} + 1}.$$
(d) (hard) The total number of electrons, \( N_e = N_d \), is shared among the donor and conduction states. Determine the chemical potential \( \mu(T) \) at low temperatures. Hint: this problem is closely related to Swendsen #28.1. Your solution will be aided if you use the identity

\[
\frac{1}{x/a + 1} + \frac{1}{a/x + 1} = 1.
\]

**Answer:** Use the given identity to replace

\[
\frac{N_d}{\frac{1}{2} e^{\beta(E_d-\mu)} + 1} \to N_d - \frac{N_d}{2 e^{-\beta(E_d-\mu)} + 1}
\]

Anticipating that \( E_d < \mu < E_c \), and taking \( \beta \) large at low \( T \), we may neglect the “+1” in the denominators to obtain

\[
\frac{N_d}{2 e^{-\beta(E_d-\mu)}} \approx \frac{N_c}{e^{\beta(E_c-\mu)}}.
\]

Finally, taking the log of both sides we solve for

\[
\mu = \frac{E_d + E_c}{2} + \frac{1}{2} k_B \ln (N_d/2N_c).
\]