Heat capacity at constant (V,N)

$C_{V,N} = \frac{\partial U}{\partial T}\Big|_{V,N}$ but... we calculate $U(\mu V T)$ not $U(N V T)$

Transformation:

$$C_N = \frac{\partial U}{\partial T}\Big|_N = \frac{\partial(U,N)}{\partial(T,N)} = \frac{\partial(U,N)/\partial(T,\mu)}{\partial(T,N)/\partial(T,\mu)}$$

"fillability"

$$= \left(\frac{\partial U}{\partial T}\Big|_\mu \frac{\partial N}{\partial \mu}\Big|_T - \frac{\partial U}{\partial \mu}\Big|_T \frac{\partial N}{\partial T}\Big|_\mu\right) \div \frac{\partial N}{\partial \mu}\Big|_T$$

$$\underbrace{=\frac{\partial U}{\partial T}\Big|_\mu}_{C_\mu} - \frac{\partial U}{\partial \mu}\Big|_T \frac{\partial N}{\partial T}\Big|_\mu \Big/ \frac{\partial N}{\partial \mu}\Big|_T$$

Equations for ideal gas

$$N = \sum_{\vec{n}} \langle n_{\vec{n}} \rangle = \sum_{\vec{n}} \left(e^{\beta(E_n-\mu)} + \sigma\right)^{-1}$$

$$U = \sum_{\vec{n}} E_{\vec{n}} \langle n_{\vec{n}} \rangle = \sum_{\vec{n}} E_{\vec{n}} \left(e^{\beta(E_n-\mu)} + \sigma\right)^{-1}$$

$$\frac{\partial N}{\partial T}\Big|_\mu = \frac{1}{k_B T^2} \sum_{\vec{n}} \frac{(E_n-\mu)\, e^{\beta(E_n-\mu)}}{\left(e^{\beta(E_n-\mu)} + \sigma\right)^2}$$

$$\frac{\partial N}{\partial \mu}\Big|_T = \frac{1}{k_B T} \sum_{\vec{n}} \frac{e^{\beta(E_n-\mu)}}{\left(e^{\beta(E_n-\mu)} + \sigma\right)^2}$$

need extra factors of $E_{\vec{n}}$ for derivatives of U.

continuum limit, as $\mu \to 0^-$

$$\frac{\partial N}{\partial \mu}\Big|_T \longrightarrow \frac{1}{k_B T}\int_0^\infty dE\, \frac{X\sqrt{E}\, e^{\beta(E-\mu)}}{\left(e^{\beta(E-\mu)}-1\right)^2} \longrightarrow (k_B T)^2 X \int_0^\infty dx\, \frac{\sqrt{x}\, e^x}{(e^x-1)^2}$$

divergent integral

$\frac{\partial N}{\partial \mu} \to \infty$ below T_{BE}

Heat capacity below T_{BE}:

$$U \sim \underbrace{\left(\frac{T}{T_{BE}}\right)^{3/2}}_{\text{fraction NOT in ground state}} \underbrace{(N k_B T)}_{\text{energy of excited states}} \Rightarrow C_{NV} = 1.925\, N k_B \left(\frac{T}{T_{BE}}\right)^{3/2}$$