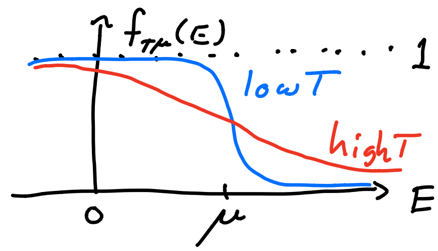
Fermi-Dirac statistics

State α energy E_α occupation $\langle n_\alpha\rangle = \underbrace{\left(e^{\beta(E_\alpha-\mu)}+1\right)^{-1}}_{f_{T\mu}(E)}$

$$N = \sum_\alpha \langle n_\alpha\rangle \to \int dE\, D(E) f_{T\mu}(E)$$

$$U = \sum_\alpha E_\alpha \langle n_\alpha\rangle \to \int dE\, E\, D(E) f_{T\mu}(E)$$



Low T, large β:

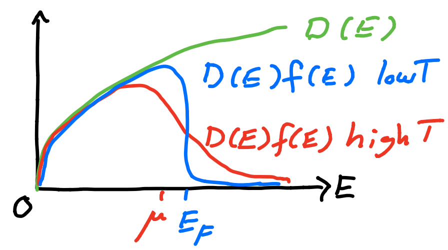
$$f \approx \begin{cases} e^{-\beta(E-\mu)} \to 0 & (E>\mu) \\ 1-e^{\beta(E-\mu)} \to 1 & (E<\mu) \\ 1/2 & (E=\mu) \end{cases} \Rightarrow \begin{array}{l} f_{T\mu}(E) \to \Theta(\mu-E) \\ \mu \to E_F \end{array} \quad \begin{array}{l}\text{step function} \\ \text{as } T\to 0\end{array}$$

"Fermi Energy"

$$N = \int_0^\infty dE\, \underset{\substack{\uparrow \\ X\sqrt{E}}}{D(E)} f_{T\mu}(E) \longrightarrow \underset{\substack{\parallel \\ \frac{2}{3} X E_F^{3/2}}}{\int_0^{E_F} dE\, D(E)}$$

ideal gas

$$X = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$$



Fermi energy E_F Fermi temperature $T_F = E_F/k_B$

ideal gas: $E_F = \frac{\hbar^2}{2m}\underbrace{\left(\frac{3\pi^2 N}{V}\right)^{2/3}}_{k_F^2}$ Fermi wavenumber

Total energy $U = \int_0^{E_F} dE\, E D(E) = \frac{2}{5} X E_F^{5/2} = \frac{3}{5} N E_F$

mean energy $= \frac{3}{5}$ maximum energy

Pressure: $P = -\frac{\partial U}{\partial V}$

$U \sim V^{-2/3} \Rightarrow P = \frac{2}{3}\frac{U}{V}$ (holds for any monatomic 3D ideal gas!) classical/quantum

Bulk modulus: $B = -V\frac{\partial P}{\partial V} = -V\left\{-\frac{2}{3}\frac{U}{V^2} + \frac{2}{3V}\underbrace{\frac{\partial U}{\partial V}}_{-P = -\frac{2}{3}\frac{U}{V}}\right\} = \frac{2}{3}\frac{U}{V} + \frac{4}{9}\frac{U}{V} = \frac{5}{3}\frac{U}{V}$