#28.1 2-level Fermi gas chemical potential

Level 0: $E_0 = 0$ Level 1: $E_1 = E$

Ω_1 —— $E_1 = 1$
Ω_0 —— $E_0 = 0$

$$N = \sum_j \Omega_j \left(e^{\beta(E_j - \mu)} + 1\right)^{-1} = \Omega_0\left(e^{-\beta\mu} + 1\right)^{-1} + \Omega_1\left(e^{\beta(E-\mu)} + 1\right)^{-1} = 1$$

note: $$\frac{1}{x+1} + \frac{1}{1/x + 1} = \frac{(1+1/x)(x+1)}{(x+1)(1/x+1)} = 1$$

1. Degeneracies $\Omega_0 = \Omega_1 = 1$ total $N = 1$ find μ, E_F

Set $x = e^{\beta\mu}$: $\Rightarrow \left(e^{\beta\mu} + 1\right)^{-1} + \left(e^{-\beta\mu} + 1\right)^{-1} = 1$ ☆

$$\Rightarrow \left(e^{\beta\mu} + 1\right)^{-1} = 1 - \left(e^{-\beta\mu} + 1\right)^{-1} \overset{\text{by } N=1}{=} \left(e^{\beta(E-\mu)} + 1\right)^{-1}$$

1 ——
$\mu = \frac{1}{2}$
0 ——

$$\Rightarrow \beta\mu = \beta(E - \mu) \Rightarrow \mu = E/2$$

2. Arbitrary Ω_0, Ω_1 with total $N = \Omega_0$: $T \to 0$

By ☆ $\Omega_0\left[1 - \left(e^{\beta\mu} + 1\right)^{-1}\right] + \Omega_1\left(e^{\beta(E-\mu)} + 1\right)^{-1} = N$ ☆☆

$\Omega_0 = N \Rightarrow \Omega_0\left(e^{\beta\mu} + 1\right)^{-1} = \Omega_1\left(e^{\beta(E-\mu)} + 1\right)^{-1}$

Set $e^{\beta\mu} \gg 1$ and $e^{\beta(E-\mu)} \gg 1 \Rightarrow \Omega_0 e^{-\beta\mu} = \Omega_1 e^{-\beta(E-\mu)}$

$$\mu = \frac{E}{2} - \frac{k_BT}{2}\ln(\Omega_1/\Omega_0) \longrightarrow \frac{E}{2} \text{ at } T = 0 \quad \therefore E_F = E/2$$

bends towards level with lower degeneracy

3. Now $N < \Omega_0$

$$\text{Rearrange ☆☆} \Rightarrow \Omega_0\left(e^{\beta\mu} + 1\right)^{-1} - \underbrace{(\Omega_0 - N)}_{>0} = \Omega_1\underbrace{\left(e^{\beta(E-\mu)} + 1\right)^{-1}}_{>0}$$

$$\text{RHS} > 0 \Rightarrow \Omega_0\left(e^{\beta\mu} + 1\right)^{-1} > \Omega_0 - N \Rightarrow \mu < k_BT\ln\frac{N}{\Omega_0 - N} \to 0$$

can show $\approx$ since $\mu \ll E$

4. Now $N > \Omega_0$: $\mu \approx E_1 - k_BT\ln\left((\Omega_0 + \Omega_1 - N)/(N - \Omega_0)\right) \to E_1$