Example : Nuclear Matter

Nuclei contain p, n both spin 1/2, similar mass $m \approx 1\,AMU$
$= 1.67 \times 10^{-27}\,kg$

Density $n \approx 0.16\,fm^{-3}$ $1\,fm = 10^{-15}\,m$

$$E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{n}{2}\right)^{2/3} = 23\ MeV$$

p or n

$$P = \frac{2}{3}\frac{U}{V} = \frac{2}{3}\cdot\frac{3}{5}\cdot\frac{N}{V} E_F = \cancel{2}\,\frac{2}{5}\,\frac{n}{\cancel{2}}\,E_F = 4\times 10^{27} \text{ atmospheres}$$

∴ "Strong" force must be very strong! Nuclei held together by gluons

Example : White dwarf star

low mass ⇒ high E_F, high P

ionized atoms = nucleons + electrons

high mass ⇒ low E_F, low P

a) Ultrarelativistic electrons : $E = \hbar k c$

DOS? $D(E) = \left(\frac{L}{2\pi}\right)^3 \int 4\pi k^2 dk\, \delta(E - \hbar k c)$

$$= \left(\frac{L}{2\pi}\right)^3 \int \frac{4\pi}{(\hbar c)^3} x^2 dx\, \delta(E - x) \qquad x = \hbar k c \quad dk = dx/\hbar c$$

$$= \left(\frac{L}{2\pi\hbar c}\right)^3 \cdot 4\pi E^2 \cdot 2 \leftarrow \text{spin } \uparrow\downarrow$$

Note: $D \sim E^2$ same as phonons, blackbody

b) $\int_0^{E_F} D(E)\,dE = N_e = \left(\frac{L}{2\pi\hbar c}\right)^3 \cdot \frac{4\pi}{3} E_F^3 \cdot 2 = \left(\frac{L}{\hbar c}\right)^3 \frac{1}{\pi^2} E_F^3$

$\Rightarrow E_F = \hbar c\, \pi^{2/3} n^{1/3}$ $n = N_e / L^3$ electron density

$$U_e = \int_0^{E_F} dE\ E\ D(E) = \frac{3}{4} N_e E_F$$

Trick: $\underbrace{\int_0^{E_F} dx \cdot x \cdot x^2}_{U_e} \Big/ \underbrace{\int_0^{E_F} dx \cdot x^2}_{N_e} = \frac{E_F^4}{4} \Big/ \frac{E_F^3}{3} = \frac{3}{4} E_F$

c) White dwarf: all $H \to He \Rightarrow \frac{1}{2} e^-/\text{nucleon}$: $N_e = \frac{1}{2} N_n$

Gravitational energy $U_G = -\frac{3}{5}\frac{GM^2}{R}$ for uniform sphere radius R

Mass M = (#nucleons)(mass/nucleon)

$= N_n m_p$

Fermi pressure vs.

Newtonian gravity: Who wins?

$$U = U_e + U_G = \frac{3}{4} \cdot N_e (\hbar c) \pi^{2/3} \left(\frac{N_e}{4\pi R^3/3}\right)^{1/3} - \frac{3}{5}\frac{G}{R}(N_n m_p)^2$$

Balance:

Fermi pressure $\approx 4.5 \times 10^{-26} N_n^{4/3}/R$

Newtonian gravity $\approx 1.1 \times 10^{-64} N_n^2/R$

Large $N_n > 2 \times 10^{57} = 1.7\, M_\odot$ gravity wins $\Rightarrow$ Collapse to neutron star

(Chandrasekhar limit)

Small N_n Fermi wins : Stabilizes at finite R due to non-ultrarelativistic electrons

Example: Neutron star

Sun-like star contains $\sim 10^{57}$ electrons, density $n \sim 10^{30} m^{-3}$

$M_\odot = 2.0 \times 10^{30}$ Kg $\quad R_\odot \sim 10^9 m \quad m_e = 9.1 \times 10^{-31}$ kg $\quad m_p = 1.7 \times 10^{-27}$ Kg

$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} = 36\, eV$ nonrelativistic $m_e c^2 = 500{,}000\, eV$

almost relativistic

White dwarf $R \to R/100 \Rightarrow n \to 10^6 \cdot n \Rightarrow E_F \to 10^4 E_F \sim 10^5\, eV$

Neutron star $R \to R/100/1000 \sim 10^4 m \quad n \to 10^9 \cdot 10^6 \cdot n$

$E_F = \pi^{2/3} \hbar c\, n^{1/3} \sim 10^8 eV$ strongly relativistic

But: inverse β-decay $p^+ + e^- + 0.8 MeV \to n + \nu_e$