Thermal Physics 2 (33-342)

Fermi-Dirac Quantum Gas Problem 1:
μ and n_0 and n_1 for a small number of particles

SOLUTION

Using the programs that you've written for quantum gases, carry out the following calculations for Fermi-Dirac statistics.

1. For the extreme cases of $N = 1, 2, 3$, and 4, compute the chemical potential μ and the occupations of the two lowest energy states as functions of the temperature.

   Don't include temperatures below $T = 0.1$ in your computations. There are numerical difficulties at very low temperatures that are not worth worrying about for this assignment.

   Go up to about $T = 3.0$ for $N = 1$, $T = 5.0$ for $N = 2$ and 3, and $T = 7.0$ for $N = 4$.

   SOLUTION:

   See the plots at the end of this document.

2. Are the Fermi energies you found in your computations consistent with what you had expected? Explain.

   #28.1

   SOLUTION: $N = \sum_j \Omega_j \left(e^{\beta(E_j - \mu)} + 1\right)^{-1}$

   Comparison with the solutions to Problem 28.1 show that they are in agreement with the Fermi energies are found from these simulations.

   For $N = 1$, the Fermi energy is halfway between $\epsilon = 3A$ and $\epsilon = 6A$.

   For $N = 2$ and $N = 3$, the Fermi energy is $\epsilon_F = 6A$.

   For $N = 4$, the ground state and the three-fold degenerate first excited state are full. The Fermi energy is halfway between $\epsilon = 6A$ and $\epsilon = 9A$. $[1^2 + 2^2 + 2^2 = 9]$

3. Explain the occupation number you found in your computations at low temperatures for a state with energy $\epsilon = 6A$ for $N = 2$ and $N = 3$.

   SOLUTION:

$E_0 = 1^2+1^2+1^2 = 3 \quad \Omega_0 = 1$

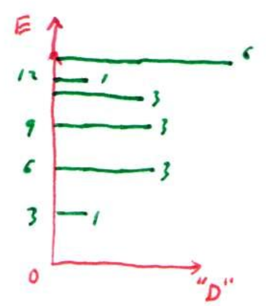
$E_1 = 1^2+1^2+2^2 = 6 \quad \Omega_1 = 3$

$E_2 = 1^2+2^2+2^2 = 9 \quad \Omega_2 = 3$

$E_3 = 1^2+1^2+3^2 = 11 \quad \Omega_3 = 3$

$E_4 = 2^2+2^2+2^2 = 12 \quad \Omega_4 = 1$

$E_5 = 1^2+2^2+3^2 = 14 \quad \Omega_5 = 6$



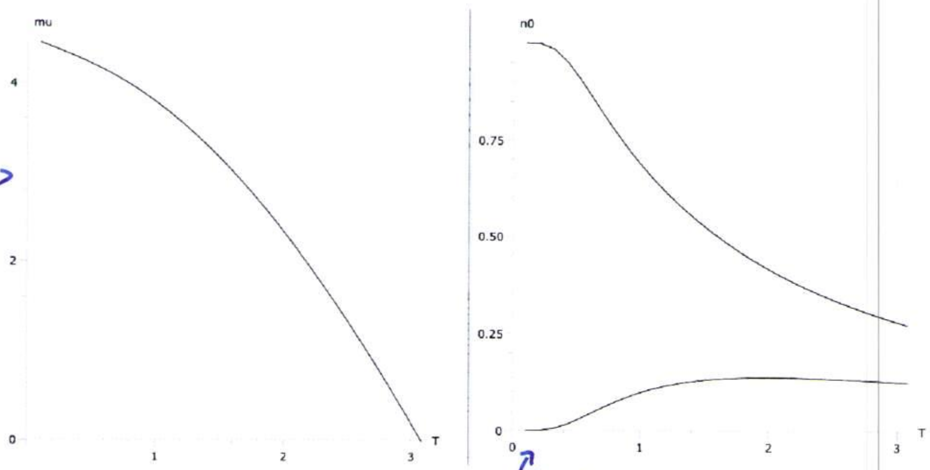
The plots and tables show that the occupations number is about 1/3 for $N = 2$ at low temperatures. This is consistent with one fermion being in energy level $\epsilon = 6A$ since this energy level is three-fold degenerate.

For $N = 3$, the occupation number at low temperatures was 2/3, which is consistent with two or the three states being occupied.

$E_F = 6 \longrightarrow$
$= 1^2+1^2+2^2$

Plots for question 1:

$E_0 = 3 \longrightarrow$
$= 1^2+1^2+1^2$



$\Big\}\ n_0, n_1 \approx 1/4$

$n_1 \to 0$

Figure 1: Plots of μ, n_0, and n_1 for Fermi-Dirac statistics with for $N = 1$ in a three-dimensional box. n_0 is the occupation of the ground state with energy $3A$, and n_1 is the occupation of the one of the three higher-energy states with energy $6A$, The Fermi energy is $\epsilon_F = 4.5$.

II 28.1
Part 2

$\Omega_0 = 1 \quad \Omega_1 = 3 \qquad N = \Omega_0 \qquad \mu = \frac{E_0 + E_1}{2} - k_B T \ln(\Omega_1/\Omega_0) + \dots$

$E_0 = 3 \quad E_1 = 6 \qquad\qquad = \frac{9}{2} - kT \ln 3 + \dots$

#28.1 part 4 $\quad \mu = E_1 - kT \ln\left(\frac{\Omega_0 + \Omega_1 - N}{N - \Omega_0}\right) + \cdots$

$N > \Omega_0 \quad \Omega_0 = 1, \Omega_1 = 3$

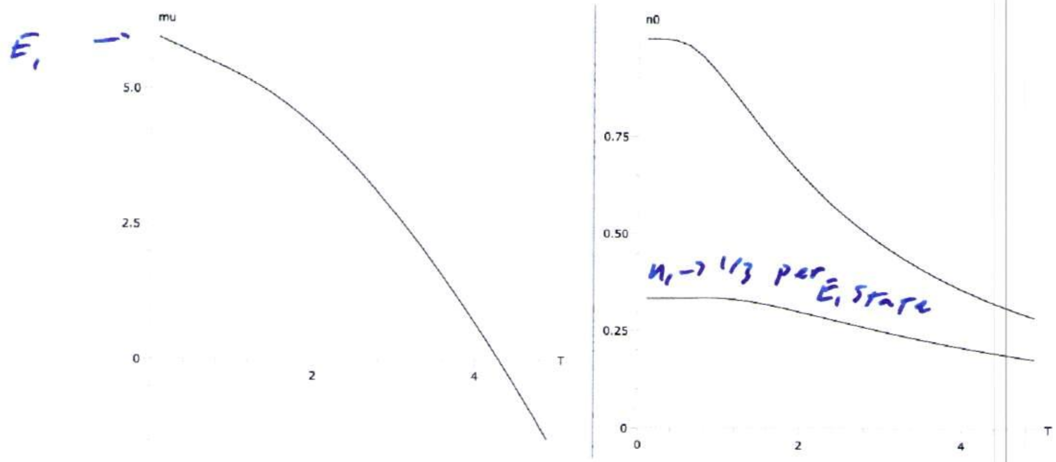


Figure 2: Plots of μ, n_0, and n_1 for Fermi-Dirac statistics with for $N = 2$ in a three-dimensional box. n_0 is the occupation of the ground state with energy $3A$, and n_1 is the occupation of the one of the three higher-energy states with energy $6A$, The Fermi energy is $\epsilon_F = 6.0$. $\quad \mu = 6 - kT \ln\left(\frac{4-2}{2-1}\right) = 6 - kT \ln 2$

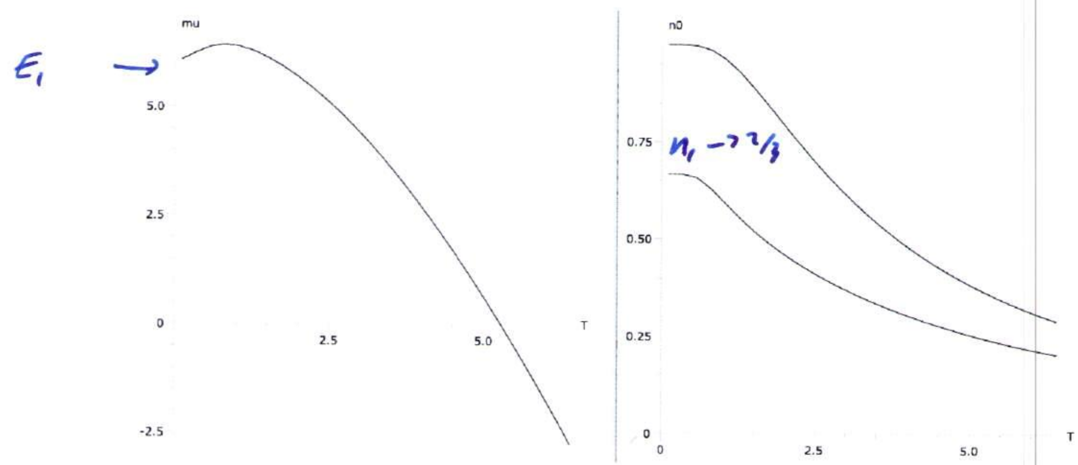


Figure 3: Plots of μ, n_0, and n_1 for Fermi-Dirac statistics with for $N = 3$ in a three-dimensional box. n_0 is the occupation of the ground state with energy $3A$, and n_1 is the occupation of the one of the three higher-energy states with energy $6A$, The Fermi energy is $\epsilon_F = 6.0$.

$\mu = 6 - kT \ln\left(\frac{4-3}{3-1}\right) = 6 + kT \ln 2$

$n_1 = 1$ per E_1 state

$\frac{E_1+E_2}{2} = \frac{15}{2} \rightarrow$

$- k_B T \ln(\Omega_2/\Omega_1)$

$\Omega_2 = \Omega_1 = 3 \Rightarrow$
$\ln(\Omega_2/\Omega_1) = 0$

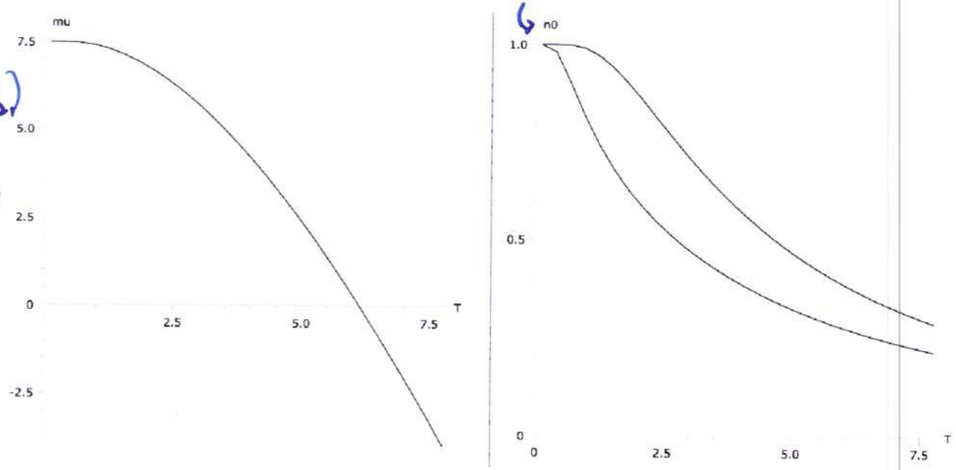


Figure 4: Plots of μ, n_0, and n_1 for Fermi-Dirac statistics with for $N = 4$ in a three-dimensional box. n_0 is the occupation of the ground state with energy $3A$, and n_1 is the occupation of the one of the three higher-energy states with energy $6A$, The Fermi energy is $\epsilon_F = 7.5$.