Models of Statistical Mechanics

1. Ising model (spin 1/2) simplified model of ferromagnet

- spins are small magnets pointing up/down ($\uparrow\downarrow$) ($+\,-$)

Magnets like to align with field $\Rightarrow$ energy $H_h = -h\sigma_j$ $\qquad \sigma_j = \pm 1$

spin at position j $\qquad$ scale out the 1/2

- Exchange interaction (quantum, electrostatic, not magnetic dipole)

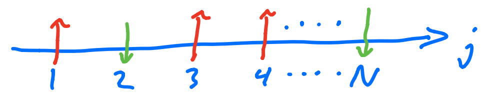
$$H_J = -J\,\sigma_i\sigma_j$$

$\langle ij \rangle$ nearest neighbor bond

$J>0$ favors $\sigma_i\sigma_j = +1$ alignment

$J<0$ favors $\sigma_i\sigma_j = -1$ antialigned

Ising Chain $H = -J\sum_{j=1}^{?}\sigma_j\sigma_{j+1} - h\sum_{j=1}^{N}\sigma_j$

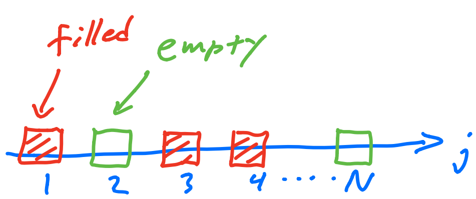


Open boundary conditions $j = 1 \cdots N-1$

periodic boundary conditions $j = 1 \cdots N$ with $\sigma_{N+1} \equiv \sigma_1$

Spin 1 Ising chain $\sigma_j = \pm 1, 0$ $\quad H = -J\sum_j \sigma_j\sigma_{j+1} - h\sum_j \sigma_j + D\sum_j \sigma_j^2$

2. Lattice gas



Define $n_j = (1+\sigma_j)/2$, $\sigma_j = 2n_j - 1$ $\qquad \sigma = -1 \Rightarrow n = 0$ "vacant"; $\quad \sigma = +1 \Rightarrow n = 1$ "occupied"

$$H = -4J\sum_{j=1}^{N} n_j n_{j+1} - (2h+4J)\sum_{j=1}^{N} n_j + N(h+J)$$

irrelevant constant

$J>0 \Rightarrow$ particles cluster

$J<0 \Rightarrow$ particles repel

$2h+4J>0$ favors occupation

$2h+4J<0$ favors vacancy

"Chemical potential"

3. Binary alloy

$$\begin{array}{cccccc} A & B & A & A & \cdots & B \\ 1 & 2 & 3 & 4 & \cdots & N \end{array} \longrightarrow j$$

$\sigma = 1 \Rightarrow A \quad \sigma = -1 \Rightarrow B$

$J > 0 \Rightarrow$ Species like themselves $\Rightarrow$ phase separation

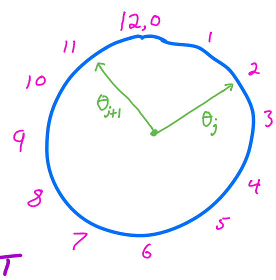
$J < 0 \Rightarrow$ mixing favored $\qquad$ field $h \sim \mu_A - \mu_B$

4. Multicomponent alloy $\sigma = 1, 2, \cdots, Q$ "Q-State Potts model"

$$H = -J \sum_j \delta_{\sigma_j \sigma_{j+1}}$$

$J > 0$ favors clusters of like-species

5. Clock model: define $\theta = \frac{2\pi}{Q}\sigma \quad \sigma = 1, 2, \cdots, Q$



$$H = -J \sum_j \underset{\uparrow}{\cos(\theta_{j+1} - \theta_j)} - h \sum_j \underset{\uparrow}{\cos(\theta_j)}$$

$J > 0$ favors alignment $\qquad h > 0$ favors $\theta = 0, 2\pi$

$\sigma = Q$

6. XY model: continuum limit of clock model $(Q \to \infty)$

$\theta \in [0, 2\pi) \qquad H = -J \sum_j \cos(\theta_{j+1} - \theta_j) - h \sum_j \cos\theta_j$

$\hat{n} = (\cos\theta, \sin\theta) \quad H = -J \sum_j \hat{n}_{j+1} \cdot \hat{n}_j \quad - h \sum_j \hat{x} \cdot \hat{n}_j$

7. Classical Heisenberg model: $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

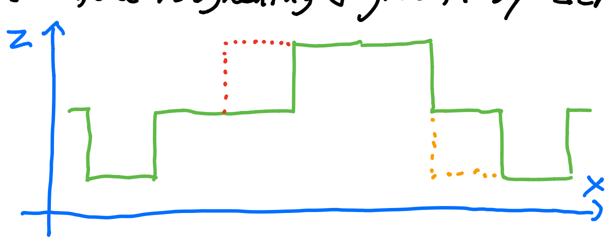
$$H = -J \sum_j \hat{n}_{j+1} \cdot \hat{n}_j - h \sum_j \hat{z} \cdot \hat{n}_j$$

8. Quantum Heisenberg model: $H = -J \sum_j \vec{S}_{j+1} \cdot \vec{S}_j - h \sum_j S_j^{(z)}$

spin operators

9. Solid-on-solid model

(surface roughening + growth by deposition: add/remove blocks



$$H = +J \sum_j (z_{j+1} - z_j)^2$$

$$\approx +J \int dx\, |\nabla z|^2 \quad \text{continuum limit}$$

$J > 0$ favors flat $\qquad T > 0$ roughens