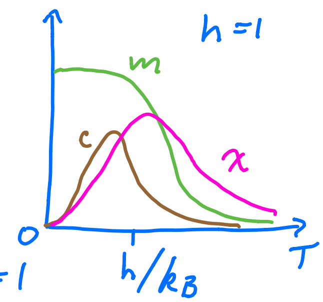
Finite Ising chain with free (open) boundaries

$$H = -J\sum_{j=1}^{N-1}\sigma_j\sigma_{j+1} - h\sum_{j=1}^{N}\sigma_j$$

special case $J=0$ $\quad H = -h\sum_j \sigma_j$ ← recall week 1!

$$Z = \sum_{\underbrace{\{\sigma\}}_{\text{all possible sets of } \{\sigma_j\}}} e^{+\beta h \sum_j \sigma_j} = \sum_{\sigma_1} e^{\beta h \sigma_1} \sum_{\sigma_2} e^{\beta h \sigma_2} \cdots \sum_{\sigma_N} e^{\beta h \sigma_N} = Z_1^N$$



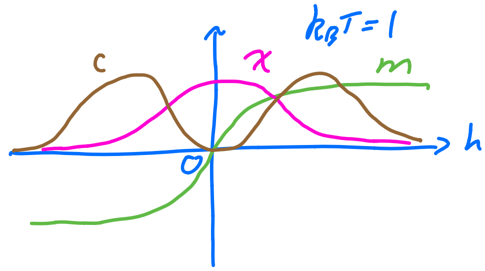
$$Z_1 = \sum_{\sigma=\pm1} e^{\beta h \sigma} = e^{\beta h} + e^{-\beta h} = 2\cosh(\beta h)$$

magnetization $m = \langle\sigma\rangle = \dfrac{e^{\beta h} - e^{-\beta h}}{Z} = \tanh(\beta h)$

$$u = \langle -h\sigma\rangle = -hm$$

$$c = \frac{\partial u}{\partial T} = \frac{h^2}{k_B T^2}\frac{1}{\cosh^2(h/k_BT)}$$

$$\chi = \frac{\partial m}{\partial h} = \frac{1}{k_BT}\frac{1}{\cosh^2(h/k_BT)}$$



Special case $h=0$:

Define $\tau_j \equiv \sigma_{j-1}\sigma_j = \pm 1 \quad (j = 2 \cdots N)$

$\vec{\sigma}$	+ − + − − + +
σ_1, τ	+ − − − + − +

$\vec{\sigma}$	− + − + + − −
σ_1, τ	− − − − + − +

$$H = -J\sum_{j=1}^{N-1}\sigma_j\sigma_{j+1} = -J\sum_{j=2}^{N}\tau_j$$

J appears as field h for spin τ!

$$Z = \sum_{\substack{\{\sigma\}\\ \sigma_1\cdots\sigma_N}} e^{-\beta H} = \underbrace{\sum_{\sigma_1}}_{1\text{ site}} \underbrace{\sum_{\{\tau\}}}_{N-1\text{ sites}} e^{-\beta H} = \underbrace{2}_{\sigma_1=\pm1} \underbrace{Z_1^{N-1}}_{\text{all } \tau} = 2(2\cosh\beta J)^{N-1}$$

$$-\beta F = \ln Z = \ln 2 + (N-1)\ln(2\cosh\beta J)$$

$$U = \frac{\partial \beta F}{\partial \beta} = -(N-1)J\tanh(\beta J)$$ ← note only $N-1$ bonds

$$c = \frac{\partial U}{\partial T} = (N-1)\left(\frac{J}{T}\right)^2\frac{1}{\cosh^2(J/T)}$$

$$S = \frac{U-F}{T} \xrightarrow[T\to0]{} k_B \ln 2$$ ← due to flipping entire chain of spins

$S/N \to 0$ in thermodynamic limit