Correlation Functions

$$H = -J\sum_{j=1}^{N}\sigma_j\sigma_{j+1} - \sum_{j=1}^{N} h_j\sigma_j \qquad PBC: \sigma_{N+1} \equiv \sigma_1$$

$$Z = \sum_{\vec{\sigma}} e^{-H} \qquad F = -\ln Z$$

note: Setting $k_B T = 1$

a) $-\partial F/\partial h_i = \partial \ln Z/\partial h_i = \frac{1}{Z}\sum_{\vec{\sigma}} \sigma_i e^{-H} = \langle \sigma_i \rangle \equiv m_i$

b) $\partial m_i/\partial h_k = \frac{1}{Z}\sum_{\vec{\sigma}}\sigma_i\sigma_k e^{-H} - \frac{1}{Z}\sum_{\vec{\sigma}}\sigma_i e^{-H}\,\frac{1}{Z}\sum_{\vec{\sigma}}\sigma_k e^{-H}$

$$= \langle\sigma_i\sigma_k\rangle - \langle\sigma_i\rangle\langle\sigma_k\rangle \equiv \chi_{ik}$$

c) Let $\vec{h} = \vec{0}$, consider translation δ: $i \to i+\delta \quad k \to k+\delta$

translational symmetry (via PBC) $\Rightarrow \chi_{i+\delta, k+\delta} = \chi_{ik}$

Let $\delta = -i \Rightarrow \chi_{ik} = \chi_{0x} \equiv g(x)$ $\quad x \equiv k - i$

$g(x)$ is a correlation function. It compares the average product $\sigma_i\sigma_k$ with the product of averages $m_i m_k$ and shows that polarizing spin i enhances polarization of spin k

d) Let $h_i = h$ be independent of i

Let $M \equiv N\cdot m = -\partial F/\partial h = \frac{1}{Z}\sum_{\vec{\sigma}}\left(\sum_i \sigma_i\right)e^{-H}$

Then $\chi_0 = \frac{1}{N}\frac{\partial M}{\partial h} = \frac{1}{N}\Big(\frac{1}{Z}\sum_{\vec{\sigma}}\sum_{ik}\sigma_i\sigma_k e^{-H}$

$$-\frac{1}{Z}\sum_{\sigma}\sum_i\sigma_i e^{-H}\cdot\frac{1}{Z}\sum_{\sigma}\sum_k\sigma_k e^{-H}\Big)$$

$$= \frac{1}{N}\sum_{ik}\chi_{ik} = \sum_{x=-N/2}^{N/2} g(x)$$