Mean Field Theory

d-dimensional cubic lattice

$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_j \sigma_j$ <ij> denote near-neighbor bond

Spins j neighboring i act as effective field on i: $h_{eff}^{(i)} = J \sum_j \sigma_j + h$ (↑ neighbors of i)

Mean (average) effective field $h_{eff} = zJ\langle\sigma\rangle + h$

- z: Coordination number, # of neighbors
- $\langle\sigma\rangle$: average polarization per spin m

Note: $h_{eff}^{(i)}$ has mean value and fluctuations

Effective single-spin Hamiltonian: $H_{\sigma_i}^{MFT} = -h_{eff}\sigma_i$

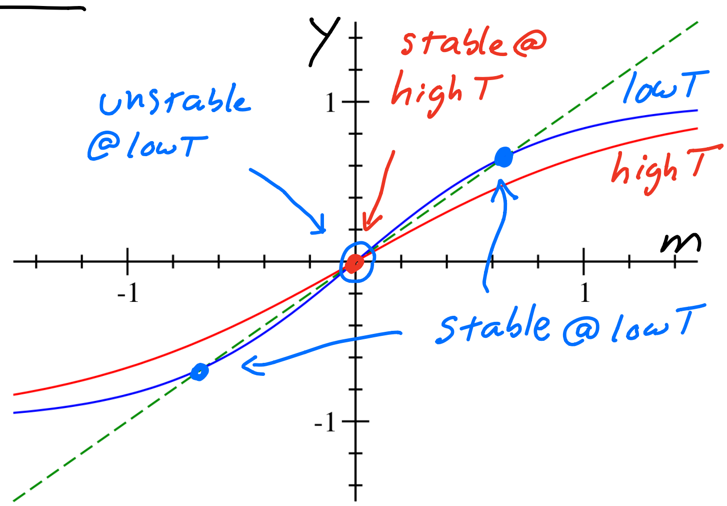
Solve for $\langle\sigma_i\rangle = \tanh(\beta h_{eff}) \Rightarrow m = \tanh[\beta(Jzm + h)]$

↑ implicit transcendental equation for m

Graphical Solution ($h=0$)

$\tanh(\beta Jzm)$ intersects $y = m$ three times at low T, one time at high T

Stable solutions are those with lowest free energy



Critical Temperature

$\tanh(\beta Jzm) = \beta Jzm - \frac{1}{3}(\beta Jzm)^3 + \cdots$

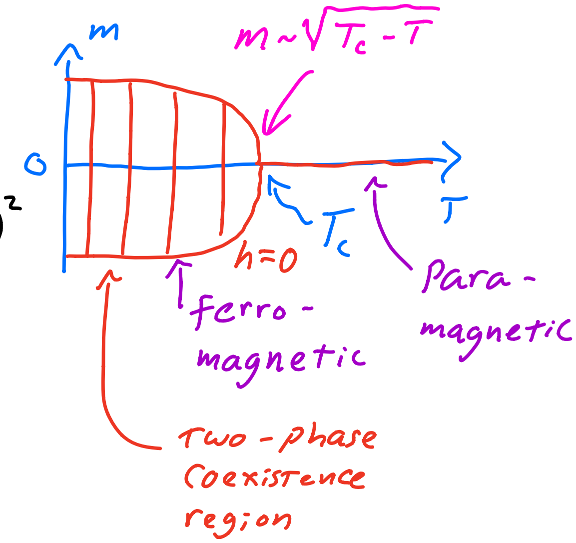
$\tanh(\beta_c Jzm)$ is tangent to $y = m$ at $\beta = \beta_c$ $\Rightarrow \beta_c Jz = 1$

$k_B T_c = Jz$

Exact solution in 2D: $k_B T_c = 2.269\,J$ $z = 4$

Numerical Solution in 3D: $k_B T_c = 4.512\,J$ $z = 6$

Magnetization below T_c



$$m = Jzm/k_BT - \frac{1}{3}(Jzm/k_BT)^3$$

$$m^2 = m^2 (T_c/T) - \frac{1}{3}(T_c/T)^3 (m^2)^2$$

$$m^2 = 3\left(\frac{T}{T_c}\right)^2 \left(1 - \frac{T}{T_c}\right)$$

$$m = \sqrt{3}\left(\frac{T}{T_c}\right)\left(1 - \frac{T}{T_c}\right)^{\beta}$$

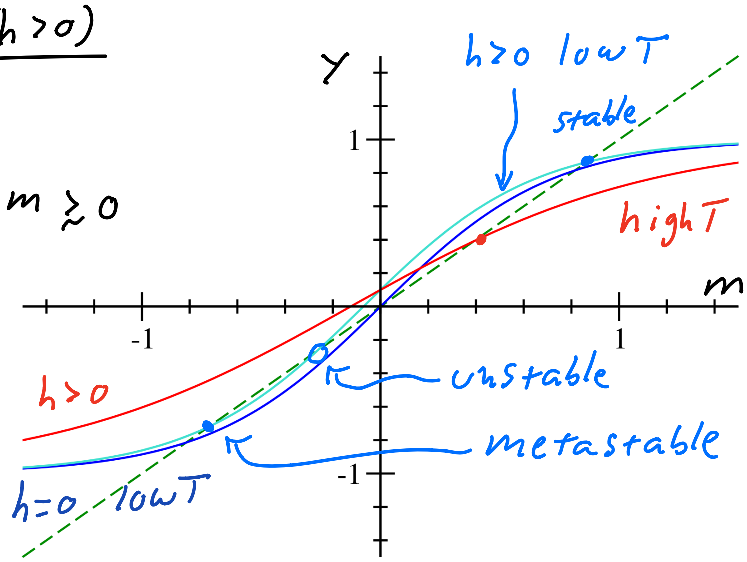
"critical exponent" $\beta = 1/2$

exact critical exponent $\beta = 1/8$ in 2D, 0.327 in 3D, 1/2 in 4D

Graphical Solution (h>0)

Weak field, $T \approx T_c$

$\Rightarrow$ Solution with small $m \gtrsim 0$

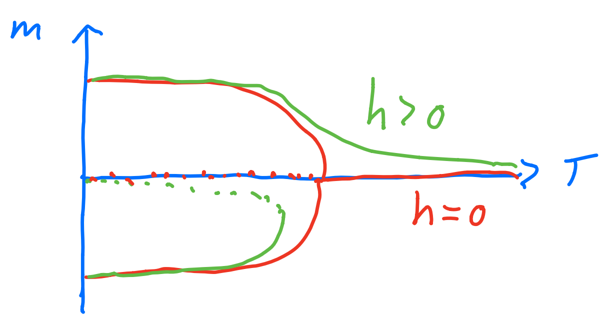


$$m = (Jzm + h)/k_BT$$

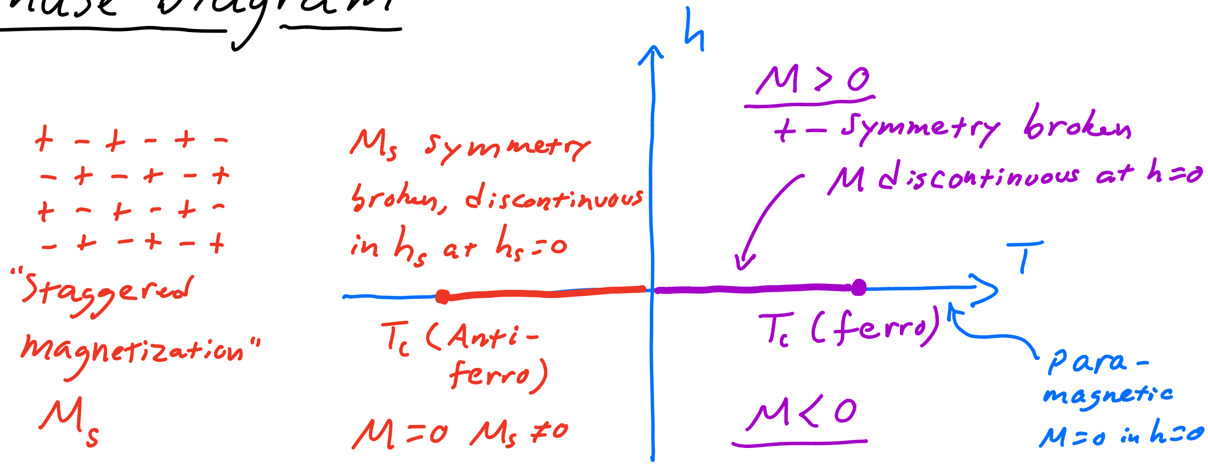
$$m\left(1 - \frac{T_c}{T}\right) = h/k_BT$$

$$m = h/k_B|T - T_c|^{\gamma}$$

$\gamma = 1$ (exact: 7/4 2D, 1.237 3D, 1 4D)



Phase Diagram



Analogy to vdWaals fluid